RECOMMENDED TERM STRUCTURE MODEL SELECTION IN A LOW-RATE MARKET
“Conscientious Choice” Ten Years Later

by Alex Levin | October 2012
# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>01</td>
</tr>
<tr>
<td>A Zero-Struck Libor Floor</td>
<td>02</td>
</tr>
<tr>
<td>Statistics of Daily Increments</td>
<td>02</td>
</tr>
<tr>
<td>Implied Volatility Skew</td>
<td>03</td>
</tr>
<tr>
<td>Dynamics of Volatility Indices</td>
<td>04</td>
</tr>
<tr>
<td>Calibrating Accuracy for At-The-Money Swaptions</td>
<td>05</td>
</tr>
<tr>
<td>Valuation of MBS</td>
<td>07</td>
</tr>
<tr>
<td>Practical Limitations of the SqG/CIR Models</td>
<td>08</td>
</tr>
<tr>
<td>Concluding Remarks and Product Information</td>
<td>09</td>
</tr>
<tr>
<td>References</td>
<td>09</td>
</tr>
</tbody>
</table>
Ten years ago, Andy Davidson asked me to back the Gaussian rate modeling choice with a market data analysis. Since the conclusion was rather evident to me, I did not expect to cause a splash when publishing the evidence. Perhaps due to the practical nature of that research and its unambiguous conclusion (“Stay Normal!”), the paper stirred interest in the mortgage-backed security community. It was immediately cited online, reproduced by the Journal of Portfolio Management [Winter 2004] and often mentioned during my conversations with clients. The paper did mention that the square-root type of volatility specification deserves attention and “may outperform others when rates are very low,” but who worried about details and nuances when rates were not so low? Back then, most Street firms used approaches of various complexities, but generally resembling the Hull-White model.

The original paper employed three types of evidence: (A) statistics of daily rate changes, (B) implied volatility skew, and (C) dynamics of volatility indices – all strongly pointing to a close-to-normal model. The paper ended: “This conclusion, however, certainly merits periodic review.” We followed our own advice and gave relevant updates in our Pipeline newsletter (December 2007, February 2011), F. Fabozzi’s Handbook of Finance [2008] and the upcoming Encyclopedia of Financial Modeling [2012]; the message generally did not change.

However, with the unprecedented fall of interest rates, the probability of getting into the negative territory using a normal distribution became too material to ignore. This paper shows that the square-root volatility pattern found in the Cox-Ingersoll-Ross (CIR) model and the Squared Gaussian (SqG) model has indeed become a better choice – until rates rise again.
A Zero-Struck Libor Floor

The 2002 paper considered the following sanity check: How large would the value of a Libor floor option struck at 0 be if we used the Hull-White (HW) model? Arguably, Libor rates (unlike Treasury rates) cannot be negative. Indeed, a bank does not need to pay another bank to store its money. It can simply store money itself or lend to its own units. A zero-struck floor is a mathematical test rather than an actual option, which should practically be worthless. Table 1 compares theoretical values delivered by the HW model in 2002 and 2012.

The model value of this worthless derivative has gone up multi-fold since 2002 due the low level of rates. Long rates and even MBS rates are not immune from getting negative within a normal model, which should cause alarm given the values in Table 1.

In this paper, we go over the three “original” tests and add (D) model calibration evidence.

### Table 1. Comparative Model Values of 0-Struck (Monthly Resetting) Libor Floors

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Libor 1M floor (in bps)</th>
<th>Libor 10-YR floor (in bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>60-mo</td>
<td>120-mo</td>
</tr>
<tr>
<td>2002</td>
<td>1.1</td>
<td>4.7</td>
</tr>
<tr>
<td>2012</td>
<td>96.2</td>
<td>170.1</td>
</tr>
</tbody>
</table>

### Statistics of Daily Increments

The first test we are going to review involves a 20-year history of daily change on the 10-yr swap rate. We collect the root-mean-squared increments and depict them against level of rates (Figure 1). Each bar in Figure 1 may contain data points from various years; the two left bars contain, of course, the recent data.

It is apparent that the relationship between absolute volatility and a rate’s level is not directional except when the rate falls very low (2.0% and lower). This may cause an issue in recommending the square root or the proportional volatility specification as a universal pattern. It seems that the desired specification should be a constant absolute level somehow connected to the origin.1

---

1The peak at the 4% rate is incidental and not related to the level of rates. This data cluster contains observations from 2008-2010 and 2003 when financial markets were volatile.
Many interest rate models can be presented in the "CEV form":

\[ dr = (\text{Drift}) dt + \sigma r^\gamma dz \]

where \( \sigma \) denotes volatility in the model-specific some notation, and \( \gamma \) represents a constant elasticity of variance (CEV). Popular short rate models are special cases of this specification: the HW model \( (\gamma = 0) \), the CIR and the SqG models \( (\gamma = 0.5) \), and lognormal models proposed by Black and Karasinski (BK) and by Black, Derman and Toy \( (\gamma = 1) \). “Unnamed” values of CEV are certainly possible including “hyper-normal” \( (\gamma < 0) \) and “hyper-lognormal” \( (\gamma > 1) \).

The market-implied CEV can be gauged from the volatility skew, i.e. dependence of Black-Scholes volatility \( \sigma_K \) on an option’s strike \( K \). Blyth and Uglum [1999] showed that, if a forward rate’s local volatility agrees with the CEV specification, then the implied volatility skew should follow the “half-degree” law:

\[ \frac{\sigma_K}{\sigma_F} = \frac{F}{K}^{\frac{1-\gamma}{2}} \]

where \( F \) is the forward rate. For example, lognormal models give us no skew; normal models depict the skew as the inverse square-root line; square-root models give an inverse fourth-degree root, etc. Hence, it is possible to measure the CEV from observed skew as we did in the 2002 paper. Back then, the optimal CEV was found to be mostly in the 0 to 0.5 range; the 1998-2002 average was 0.23, but it became closer to zero later. Figure 2 shows that the skew measured recently for the 5-into-10 swaption follows the CEV of 0.5 very closely and justifies the use of the CIR/SqG models.
To compare rate models and to attribute changes of the market, it is useful to design a market volatility index – a single number reflecting the overall level of option volatility deemed relevant to the interest-rate market. The volatility index can be defined as a short-rate volatility constant that (coupled with a zero mean reversion) best approximates the ATM swaption volatility surface. One can roughly view the index as an average of market volatility, expressed in a model-specific form: the absolute form for the HW model, the proportional form for the BK model, etc. In non-linear models, however, the short rate local volatility is not equal to the Black volatility, and the volatility index is not exactly equal to the average market volatility, even in the complete absence of mean reversion.

Instability of the volatility index may serve a sign that model’s specification is directionally biased. For example, as seen in Figure 3, the dynamics of the BK index (red line) is mirror reflective of the level of interest rates. This observation suggests that a proportional volatility cannot stay constant when rates move up or down.

To make the directionality (or its absence) apparent, we show indices and rates relative to their respective Jan-2000 levels. We see that the HW volatility index (green) has been the most stable one overall except when rates were very low. Not surprisingly, the SqG index (blue) has been the most stable lately.

Dynamics of Volatility Indices
This is a measure pointing to the accuracy of a model's ability to reproduce at-the-money (ATM) market volatilities on various tenors simultaneously. In order to explain this test, consider just a pair of options, e.g. 1-year options on a 2-year swap and a 10-year swap. Having a pair of free-to-choose parameters, volatility constant $\sigma$ and mean reversion $\alpha$, we are able to match the two options' prices precisely. By allowing $\sigma(t)$ and $\alpha(t)$ to be functions of time, we can match prices of many pairs of swaptions with all expirations. The question arising during this calibration process is whether the result is reasonable. For example, let us say we have a steep yield curve and the 1-year ATM options' volatilities are 80% for a 2-yr swap and 40% for a 10-year swap. If we use the Black-Karasinski model, we can explain the difference between two options by a brutal force of mean reversion. This blind approach would result in an unrealistically large value of $\alpha$. Let us imagine further that the forward rates are 1% for the 2-yr swap and 2% for the 10-year swap thereby making the absolute volatilities for the two tenors identical at 80 bps. Should we use the HW model with $\alpha = 0$, these two options would be perfectly matched.

Let us modify this example and change the forward rate from 2% to 2.5% for the 10-year swap thereby making its absolute volatility 100 bps, i.e. higher than absolute volatility of the 2-year swap. It would now require us to select a negative mean reversion for the HW model to explain why volatility increases with maturity. In turn, a negative mean reversion causes instability and does not serve as a realistic choice for a stable system. The 80/100 bps volatility structure could be explained by shifting away from the HW to the SqG/CIR model or a two-factor model, for example.
As seen from these cases, with a steep yield curve, one can test how well options on different swaps are replicated within a single-factor model. Our term-structure modeling library is equipped with a fast pseudo-analytical calibrator, working very accurately for the HW model and approximately for the SqG model and the BK model. It usually takes just 0.5 to 1.0 sec to calibrate the model while running the first instrument in a batch and does not involve explicit option pricing. Since we allow the volatility function to depend on time and use a constant, non-negative, mean reversion, swaptions are matched on average. Figures 4, 5, and 6 depict the result of this process as we compare different models.

**Figure 4: The HW model (RMSE = 13.4 bp)**
Calibration to ATM Swaptions as of August 31, 2012 (bars = market; lines = model)

**Figure 5: The SqG Model (RMSE = 2.2 bp)**
Calibration to ATM Swaptions as of August 31, 2012 (bars = market; lines = model)
On this comparative basis, the HW model is now not accurate as yielding the largest RMSE and missing the difference between short-dated options on 2-year and 10-year swaps. The SqG model and the BK model are much more accurate with the BK model working better for shorter-dated (and less relevant) expiries and worse for others, but requiring an unrealistically looking $\sigma(t)$.

### Valuation of MBS

Our 2002 paper explained how the interest rate modeling choice affects valuation measures. Quite simply, for TBAs, a higher CEV index causes a longer option-adjusted duration (OAD). This conclusion follows the fact that TBAs have negative Vegas. When measuring OAD, we stress rates up and down and each move up causes absolute volatility to grow in proportion to $r^\gamma$. An increase of volatility reduces the value thereby inflating the overall exposure. This is an indirect volatility specification effect on OAD that differentiates one model from another. In 2002, we estimated, the HW model was “shorter” than the BK model by about 0.3-0.5 yr, on an OAD basis (with the SqG model being in between). However, with rates as low as they are now, this difference has grown larger.

As for the OAS differences, speaking generally, a higher CEV will cause lower OAS levels for high premium TBAs and higher OAS levels for the par-coupon and discount TBAs. Given that the entire market is at premium, the first part of the statement applies. In order to explain this sensitivity pattern, let us ask ourselves the following question: What should happen with interest rates in order for prepayment option embedded in the TBA to be at-the-money? For a high-premium TBA, rates should move up inflating volatility as $r^\gamma$. Note that the ATM point is where the option’s Vega is the largest. Once this is established, we can conclude that a larger CEV causes the prepay option’s value to be larger. For par-coupon and discount TBAs, the result is the opposite.
Table 2 compares AD&Co’s OAS and OAD under the two models.

Although OAS and OAD metrics depend severely on the prepayment model used, the SqG-generated OAD column in Table 2 is closer to the dealers’ median than the HW-generated one.

<table>
<thead>
<tr>
<th>Coupon</th>
<th>HW OAD</th>
<th>HW OAS</th>
<th>SqG OAD</th>
<th>SqG OAS</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4.04</td>
<td>0</td>
<td>5.12</td>
<td>3</td>
</tr>
<tr>
<td>3.5</td>
<td>1.84</td>
<td>-11</td>
<td>2.54</td>
<td>-16</td>
</tr>
<tr>
<td>4</td>
<td>0.15</td>
<td>-24</td>
<td>0.53</td>
<td>-36</td>
</tr>
<tr>
<td>4.5</td>
<td>-0.03</td>
<td>-41</td>
<td>0.24</td>
<td>-55</td>
</tr>
<tr>
<td>5</td>
<td>1.25</td>
<td>-31</td>
<td>1.5</td>
<td>-42</td>
</tr>
<tr>
<td>5.5</td>
<td>2.57</td>
<td>81</td>
<td>2.87</td>
<td>73</td>
</tr>
<tr>
<td>6</td>
<td>2.73</td>
<td>129</td>
<td>3.03</td>
<td>121</td>
</tr>
</tbody>
</table>

Practical Limitations of the SqG/CIR Models

Despite evidence, analysts and developers alike may face some practical limitations when engineering or utilizing the most suitable rate square-root volatility models.

1. A yield curve cannot have negative forward points. This restriction applies to both the actual curve and curves stressed down for Duration/Convexity computations. As a result, the shocks utilized for computations may have to be smaller than standard or desired.

2. With a typical implementation on a sparse rate tree, grid, or with finite time steps, close-to-zero rate points cannot be accurately matched. For example, in the SqG model, the short rate \( r = x^2 \) where \( x \) is a normal deviate. Expectation of \( r \) computed on a discrete grid of values is going to be greater than some positive minimum.
Concluding Remarks and Product Information

The square root volatility specification matches the current market well, but this match seems to be limited to the very low rate environment we are in. Once again, “this conclusion merits periodic review.”

The Squared Gaussian model recommended in this paper is available via AD&Co’s suite of interest-rate models, AD&Co’s OAS model, and the end-user valuation product, RiskProfiler™. Current clients may contact AD&Co before using the SqG model for additional information and necessary upgrades.

In addition, AD&Co’s weekly MBS market analyses are now available on www.ad-co.com with the SqG version along with the HW version.

References


Acknowledgements

The author would like to thank Andrew Davidson who proposed conducting the study and Nancy Davidson and Simone Davis for editorial and publishing efforts.
Quantitative Perspectives is available via www.ad-co.com

We welcome your comments and suggestions.

Contents set forth from sources deemed reliable; but Andrew Davidson & Co., Inc. does not guarantee its accuracy. Our conclusions are general in nature and are not intended for use as specific trade recommendations.

© Copyright 2012 Andrew Davidson & Co., Inc.