QUANTITATIVE PERSPECTIVES

DIVIDE AND CONQUER: EXPLORING NEW OAS HORIZONS

By ALEX LEVIN

PART I: ACTIVE-PASSIVE DECOMPOSITION, September 2003
PART II: A PREPAY RISK-AND-OPTION-ADJUSTED VALUATION CONCEPT, March 2004
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ANDREW DAVIDSON & CO., INC.

DIVIDE & CONQUER: EXPLORING NEW OAS HORIZONS

PART I: ACTIVE-PASSIVE DECOMPOSITION

SEPTEMBER 2003

by Alex Levin

Introduction

In two previous publications (Levin [2001, 2002]), a new method of mortgage valuation modeling called active-passive decomposition (APD) was introduced. An efficient alternative to brute-force Monte-Carlo, the APD method splits a mortgage pass-through into two path-independent components, the active (refinancable) and the passive (non-refinancable). Once this is done, the most time-efficient pricing structures operating backwards on probability trees or finite-difference grids can be employed. This valuation method runs faster than Monte-Carlo while delivering a much richer outcome — all stressed values required by mandatory risk assessments — at no additional cost. Risk managers and traders of unstructured mortgage instruments, such as agency pass-through MBS, whole loans, stripped (IO/PO) derivatives, including mortgage servicing rights (MSRs) may directly benefit from the method.

The APD approach simulates the burnout effect in a natural and explicit way - through modeling the heterogeneity of the collateral. Hence, it presents an analytical advantage over any other approach that requires ad-hoc judgments about the achieved degree of burnout. Structured instruments, CMOs and ABS, though they retain heavy sources of path-dependence (other than the burnout) and still rely on Monte-Carlo pricing, can gain from a better, more robust prepay modeling.

The multi-population view of mortgage collateral is a known approach to explain the burnout effect. In one of the earliest modeling attempts, Davidson [1987] and Davidson et al [1988] proposed the Refinancing Threshold model in which collateral is split into three or more American option bonds having differing strikes. A conceptually similar approach proposed recently by Kalotay and Young [2002] divides collateral into bonds differing by their exercise timing. Such structures naturally call for the backward induction pricing, but they fall short in replicating actually observed, probabilistically smoothed prepayment behavior - even if many constituent bonds are used. On the other hand, contemporary analytical systems used by the Street often employ multi-population mortgage models (see Hayre [1994, 2000]), but do not seek any computational benefits, as they rely heavily on Monte-Carlo pricing anyway.

The APD is a "mortgage-like" model, with a refinancing S-curve, aging, and other ad-hoc features that are meant to capture non-efficient, empirical option exercise. Therefore, it is capable of generating realistic prepayment behavior with only two constituent components, the active and the passive. We will give an overview of the extended APD model in Part I of this *Quantitative Perspectives* series. In this extended model, we allow passive mortgagors to be partial refinancers, albeit at a reduced speed. Such an extension, while not elevating technical complexity of the model, leads to a better historical fit, especially for very burnt pools.

Because the model relies on the backward induction valuation performed separately for its two components, it naturally assesses their future prices - very much unlike Monte-Carlo. It sets an analytical framework where both valuation and prepayments can be modeled rigorously, beyond the traditional OAS framework. We will present an elegant CAPM-like mortgage valuation model in the Part II of this series. In this model, armed with the power of backward induction, we endogenously find prepay risk measures and the price of risk in the form of return spread compensation for each investment period and level of interest rates. A value thus obtained, therefore, will carry financially well-defined return demand and may operate without an exogenous OAS input. The entire valuation model implemented on the same probability tree or finite-difference grid will deliver objective economic measures (price, OAS, duration, convexity, option cost, etc.) from a sector-wide spread measure, such as the agency debt spread. For example, values for IOs, POs and MSRs can be objectively derived without knowing the differences in OAS; OAS, in fact can be calculated from these economic values once they are determined.

Path-dependence & Pricing PDE

Let us consider a hypothetical dynamic asset ("mortgage") market price of which $P(t,x)$ depends on time t and a single market factor x. The latter can be formally anything and does not necessarily have to be the short market rate or the yield on the security analyzed. We treat $x(t)$ as a random process having a (generally variable) drift rate μ and a volatility rate σ and being disturbed by a standard Brownian motion $z(t)$, i.e.

$$dx = \mu dt + \sigma dz \tag{1}$$

We assume further that the asset continuously pays the $c(t,x)$ coupon rate and its balance B gets amortized at the $\lambda(t,x)$ rate, that is $\frac{\partial B}{\partial t} = -\lambda B$. One can then prove that the price function $P(t,x)$ should solve the following partial differential equation (PDE):

$$\underbrace{r + OAS}_{\text{expected return}} = \underbrace{\frac{1}{P}\frac{\partial P}{\partial t} + \frac{1}{P}(c+\lambda) - \lambda}_{\text{time return}} + \underbrace{\frac{1}{P}\frac{\partial P}{\partial x}\mu}_{\text{drift return}} + \underbrace{\frac{1}{2P}\frac{\partial^2 P}{\partial x^2}\sigma^2}_{\text{diffusion return}} \tag{2}$$

A derivation of this PDE can be found in Levin [1998], but it goes back at least to F. Fabozzi and G. Fong [1994]. A notable feature of the above written PDE is the absence of the balance variable, B. The entire effect of possibly random prepayments is represented by the amortization rate function, $\lambda(t,x)$. Although the total cashflow observed for each accrual period does depend on the beginning-period balance, construction of a finite difference scheme and the backward induction will require the knowledge of $\lambda(t,x)$, not the balance. This observation agrees with a trivial practical rule stating that relative price is generally independent of the investment size.

Pricing PDE (2) can be solved on a probability tree or finite difference pricing grid that has as many dimensions as the total number of factors or state variables that affect r, c, and λ. In particular, if the coupon rate is fixed, and the amortization rate λ depends only on current time (loan age) and the immediate single market factor x, the entire valuation problem can be solved backwards on a two-dimensional (x,t) lattice. To implement this method, we would start our valuation process from maturity T when we are sure that the price is par, $P(T,x)=1$, regardless of the value of factor x.

Working backwards, we derive prices at age t-1 from prices already found at age t. In doing so, we replace derivatives in PDE (2) by finite difference approximations, or weigh branches of the lattice by explicitly computed probabilities. If the market is multi-factor, then x should be considered a vector; the lattice will require more dimensions. Generally, the efficiency of finite-difference methods deteriorates quickly on high-dimensional grids because the number of nodes and cash flows grow geometrically; probability trees may maintain their speed, but at the cost of accuracy, if the same number of emanating nodes is used to capture multi-factor dynamics. If we decide to operate on a probability tree instead of employing a finite-difference grid, then for every branch:

$$P_k = \frac{c_k + P_{k+1} + \lambda_k (1 - P_{k+1})}{1 + r_k + OAS} \qquad (3)$$

where P_k is the previous-node value deduced from the next-node value P_{k+1}. Probability weighting of thus obtained values applies to all emanating branches.

Extended Active-Passive Decomposition Model

The Concept

Even for a simple fixed-rate mortgage pass-through, total amortization speed λ cannot be modeled as a function of time and the immediate market. Prepayment burnout is a strong source of path-dependence because the future refinancing activity is affected by past incentives. One can think of a mortgage pool as a heterogeneous population of participants having different refinancing propensities. Some mortgagors have higher rates, better credit, larger loans, or perhaps they face smaller state-enforced transaction costs. Once they leave the pool, the future prepayment activity gradually declines.

Instead of considering pricing PDE for the entire collateral, we propose decomposing it first into two components, "active" and "passive," differing in refinancability. Under the following two conditions, mortgage path-dependent collateral can be deemed a simple portfolio of two path-independent instruments:

(A) Active and passive components prepay differently, but follow the immediate market and loan age.

(B) Any migration between components is prohibited.

The Details

Here is an example that fits the above criteria:

$$\begin{aligned} \text{ActiveSMM} &= \text{RefiSMM} + \text{TurnoverSMM} \\ \text{PassiveSMM} &= \beta * \text{RefiSMM} + \text{TurnoverSMM} \end{aligned} \tag{4}$$

where RefiSMM denotes active speed and TurnoverSMM is the turnover speed, both are assumed to depend on market rates and loan age only. Parameter β quantifies relative refinancing activity for the passive component; it takes values between 0 and 1.

In order to find the total speed, we have to know the collateral composition. Denote ψ the ratio of active group to total, then

$$\lambda \equiv \text{TotalSMM} = \psi * \text{ActiveSMM} + (1 - \psi) * \text{PassiveSMM} \tag{5}$$

All variables are time-dependent, but we omitted subscript t for simplicity. The initial value of ψ describes composition of collateral at origination; both ψ_0 and β are parameters for a particular prepay model. Dynamic evolution of ψ from one time moment (t) to the next ($t+1$) is as follows:

$$\psi_{t+1} = \psi_t \frac{1 - ActiveSMM_t}{1 - TotalSMM_t} \tag{6}$$

It is worth considering a few special cases. First, if ψ is zero at any instance of time, it will remain zero for life. Second, if ψ is 1 at any time, then it will retain this value as well because TotalSMM is identical to ActiveSMM from equation (5). Indeed, if the mortgage pool is either totally passive ($\psi=0$) or totally active ($\psi=1$), it will retain its status due to complete absence of migration. In either of these two special cases, variables ψ and TotalSMM are path-independent, leading us to a key conclusion: considering active and passive components separately avoids the problem of path-dependence altogether.

How the Model Works Forward

If $0<\psi<1$, then TotalSMM < ActiveSMM, the fraction in the right-hand side of formula (6) is less the 1, and ψ gradually falls. If we employed the APD model for prepay modeling while using Monte-Carlo for valuation, we could innovate the compositional variable ψ month after month. First, we would compute refinancing and turnover speeds at time *t* from their respective models. Then, we would produce active, passive and total speeds, all still at time *t*, from formulas (4) and (5). This information is sufficient to generate the *t*-month cash flow and also allows for finding the next-month composition, ψ_{t+1}, from formula (6) and proceed forward.

Note that prepay speeds RefiSMM and TurnoverSMM depend only on current market rates and time, i.e. they are path-independent. Naturally, ActiveSMM and PassiveSMM found from (4) will be path-independent as well. In contrast, variables ψ and TotalSMM are generally path-dependent — except when ψ is either 0 or 1.

Let us look at how the APD model works. Suppose we have a pool with $\psi_0=0.8$, i.e. the active part constitutes 80% of total, at origination. Consider two possible scenarios: (A) rates drop and remain low inducing refinancing activity, and (B) rates rise and remain high. Figures 1A & 1B show how the pool composition will evolve in these two cases.

Figure 1A:

Rate Decrease (refinancing wave)

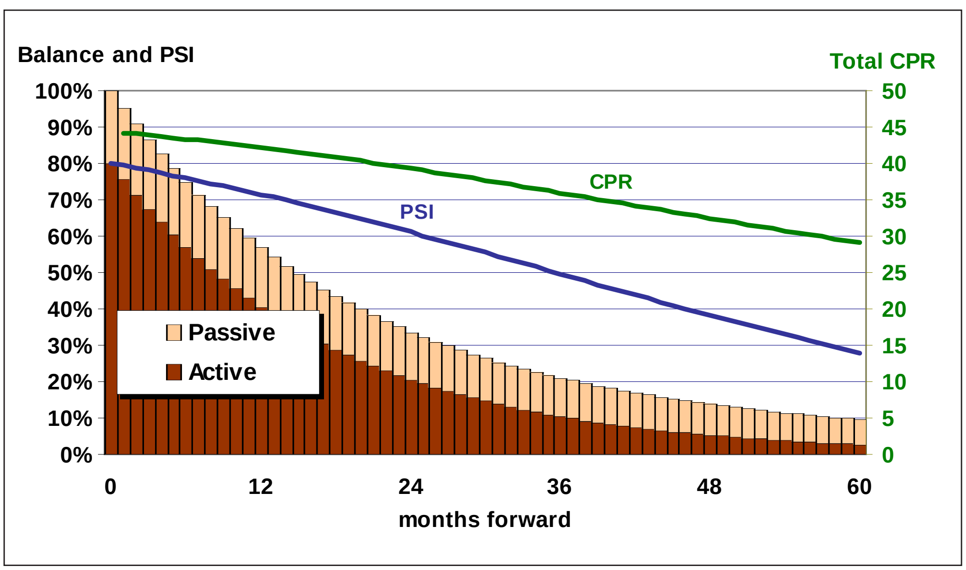
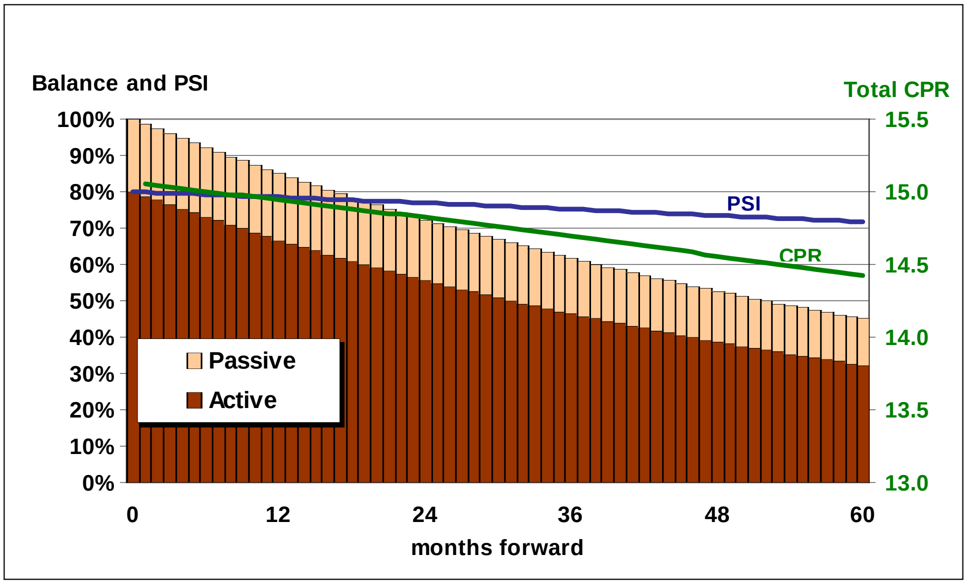


Figure 1B:

Rate Increase



For scenario (A), pool balance gets amortized quickly due to the refinancing wave, but, more importantly, the active group (brown bars) evaporates much faster than the passive group (beige bars). As a result, variable ψ (blue line) drops from the original 80% to under 30%, and, correspondingly, the total speed (green line) declines — in the complete absence of any rate dynamics. A sizable speed reduction from 45 CPR to 30 CPR is caused exclusively by the burnout effect and reflected by ψ. This effect is not seen on Figure 1B, where the active and the passive groups retire at similar rates. Pool composition barely changes, as does the total prepayment speed.

We could give prepayment behaviors seen in Figures 1A and 1B another interesting practical interpretation. Let us assume that we wish to compare a regular fixed-rate pool (Figure 1A) with a prepayment-penalty pool (Figure 1B) under the same, low-rate market conditions. The regular pool burns out — unlike the prepay-penalty one, which faces additional refinancing barrier. At the end of its penalty window (assume 60 months), this pool retains a relatively high level of ψ (71.7%). Looking at a matching speed level in Figure 1A, we conclude that, once the penalty window is over, the prepay speed will jump above 40 CPR (compared to 29 CPR of the regular pool). Therefore, the APD model naturally explains the "catch-up" effect actually known for prepay-penalty mortgages.

Above, we assumed a newly originated pool, the population of which is determined by parameter ψ_0. In practice, a pool may be already seasoned, and today's value of ψ, denote it $\psi(t_0)$, needs to be determined first. We will cover this task shortly.

How the Model Works in Backward Induction

If we decide to employ the APD model for backward valuation, we do not need to innovate path-dependent variables, ψ and TotalSMM, or keep track of their dynamics. Below are a few simple steps to perform:

Step 1 Recover today's value of the population variable, $\psi(t_0)$.

Step 2A Generate cash flows on each node of a pricing grid (tree) for the active part only, and value it using a backward inducting scheme that solves pricing equation (2).

Step 2B Do the same for the passive part.

Step 3. Combine thus obtained values as:

$$P = \psi(t_0)P_{active} + [1-\psi(t_0)]P_{passive} \tag{7}$$

Interestingly enough, formula (7) applies to today's prices obtained for all interest rate levels of the pricing grid. As stated above, computing prices on the entire grid is an inseparable part of backward valuation. Therefore, the total price can be also found on the grid at no additional cost.

In particular, the Greeks (e.g., durations, convexity) are found immediately, without any repetitive efforts with stressed market (compare with Monte-Carlo!) However, we cannot apply formula (7) for future nodes because we know only $\psi(t_0)$ — today's value of ψ.

Initializing the Burnout Factor

If the pool is already seasoned, we have to assess $\psi(t_0)$ first before we can employ the APD model either for forward simulation or backward induction. Two main approaches may be employed to solve this problem, an analytical closed-form method, or historical simulations.

Let us suppose we know the pool's age, t_0, factor, $F(t_0)$, and a constant turnover rate[1], $\lambda_{turnover}$. We can then assess the turnover factor $F_{turnover}(t_0)=\exp(-\lambda_{turnover}t_0)$ along with the scheduled factor, $F_{scheduled}(t_0)$. Since the entire pool's amortization is driven by refinancing, turnover, and the scheduled payoff, knowing two out of three factors along with the total pool's factor is enough to restore the entire time t_0

[1]We can relax this condition by assuming turnover rate is known, not necessarily constant.

composition. It is easy to show that unknown $x=\psi(t_0)$ satisfies the following, generally transcendent, algebraic equation:

$$x+\alpha x^{\beta}=1 \tag{8}$$

where α is a known parameter:

$$\alpha=\frac{1-\psi_0}{\psi_0^{\beta}}\left[\frac{F_{turnover}(t_0)F_{scheduled}(t_0)}{F(t_0)}\right]^{1-\beta}$$

and β is the same speed-reducing multiplier that enters the APD model (4).

Of course, no numerical iterations are needed if β is 0, 1, or 0.5. For instance, β=1 is a trivial case when the pool is homogeneous and is not subject to burnout, $\psi(t_0)=\psi_0$. Case β=0 was considered in Levin [2001, 2002]; it leads to:

$$\psi(t_0)=1-(1-\psi_0)\frac{F_{turnover}(t_0)F_{scheduled}(t_0)}{F(t_0)}$$

A simple quadratic equation for $\psi(t_0)$ arises when β=0.5, with only one meaningful positive solution. For all other values of β, numerical methods will suffice.

Solving equation (8) is an attractive way to initialize the burnout stage, as it does not require a historical simulation of past refinancing incentives. However, it is valid only for the very specific form of the APD model presented by formulas (4) and (5). Any possible extension of the model (such as discussed below) will make it impossible to recover the burnout stage using the pool's factor and age information, only. An alternative method to estimate $\psi(t_0)$ would be a historical simulation of all prepayment components, i.e. running the APD model forward, from a pool origination until today. A relevant historical interest rate database will be required to facilitate this process, which constitutes the essence of AD&Co.'s data files currently distributed to clients, monthly.

Extensions & Nuances

More components, more prepay sources

The APD model (4)-(6) is a two-component pool model exposed to two sources of prepayment, refinancing and turnover. Each of these features prepayment patterns (super-active, active, moderately active, etc.). On the other hand, there may exist prepayment sources that contribute to each of the groups, but are distinctly different from refinancing and turnover. Let us briefly review both ways to extend the model.

As we already pointed out, even a two-component model ensures smooth prepayment behavior if each component does so. Within our APD framework, a refinancing model may include a traditional S-like curve, aging and perhaps some other known empirical mortgage effects that can be attributed to a non-optimal option exercise. The total prepayment speed is proven to be between RefiSMM and TurnoverSMM, being continuously weighted as controlled by variable $\psi(t)$. Adding more components into the model does not alter this fact, nor does it add any smoothness to the prepay model. It is also more difficult to fit a three- or four-component model than the APD model presented here.

The APD model (4), (5) assumes that the active and passive components share the same turnover rate, and their refinancing speeds relate to one another as 1 to β. We can consider some other prepayment source that is not propagated to the active and passive components identically, or with the 1 to β ratio. For example, we may introduce a credit cure prepay source, additional to refinancing and turnover, but likely having higher effect on the passive part than on the active part. The cash-out refinancing driven by home prices carries a similar effect.

Of course, additional prepayment sources can be formally included in the refinancing without assuming any longer that active and passive refinancing models relate to one another as 1 to β. We will not be able to initialize $\psi(t_0)$ by solving equation (8), and we must use historical simulations for this purpose as discussed above. Principally, we may assume unrelated refinancing models built for the active and passive components, gaining generality with little sacrifice of convenience.

Modeling Refinancing Speed as a Function of Price

Having asserted that the APD model is a "mortgage-like" model, we mean to distinguish it from the other popular academic approach, the rational prepayment exercise models (see Longstaff [2003], Stanton [1995]). Yet, the APD model can address some shortcomings typically known for purely ad-hoc experimental models. As we have already affirmed, the APD model can value MBS backward, provided that its refinancing and turnover constituents depend only on the current market. A likely implementation of this rule would rely on some experimental relationship between the SMMs and a relevant mortgage index. Although this is the way most mortgage practitioners envision prepayment modeling, it is not the only possible approach. In fact, refinancing behavior of homeowners also depends on the type of mortgage in hand. Given the coupon and market, economic incentive to prepay vanishes when maturity, balloon or ARM reset approach.

An attractive alternative would be linking refinancing speeds of a mortgage (still measured on the grid nodes, separately for the active and passive pieces) directly to its price appreciation using path-independent specification RefiSMM(Price) instead of RefiSMM(Rate). This is the same approach used for the valuation of American option bonds, except the refinancing model can still be an experimental S-curve, not the "optimal" or "rational" exercise rule. This model would state the refinancing speed, RefiSMM, as a function of price premium, for example, 2 SMM if collateral is priced at 102, 6 SMM for 104, etc., asymptomatically approaching its "ultimate" speed. Formulas (4), (5) still allow for computing the active, passive, and total speeds. In particular, the passive component will still refinance at a beta-reduced speed for the same price premium as the active component.

Such a refinancing model will still be path-independent, presenting no theoretical or computational issues for the backward valuation. Moreover, if the refinancing behavior is indeed driven by price appreciation and such a universal relationship can be experimentally established, then the APD modeling approach and its backward implementation becomes a natural, if not only, way to price an MBS. In contrast, Monte-Carlo-based valuation method simply would not allow assessing future prices and, hence, prepayment speeds.

Arguably, the RefiSMM(Price) function can be viewed as one universal refinancing rule that can serve many collateral types. Furthermore, such a model can directly account for additional, loan-specific, transaction cost and cost-saving opportunities. For example, the knowledge of prepayment penalties, average loan sizes, or state-imposed taxes can easily be used to modify the S-curve.

Valuation Features of Mortgage Servicing Rights (MSR)

FAS 133 regulation has increased the practical importance of quick yet rigorous valuation of MSR. Now, managers of MSRs have to mark to market their assets often and monitor values against fairly complex hedges, both mandatory and optional. As explained in Levin [2002], MSR can benefit from the use of the APD model. Here is the main argument.

MSR are different from an IO in that they carry some fixed (non-proportional) dollar income and cost components counted per loan. For example, a mortgage servicer may receive an annual $40 per loan in the form of ancillary income (floats, insurance fees, etc.) -- regardless of the loan size. It is clear that the proportional rate c used in pricing PDE (2) will now change gradually with the average loan balance, even if the stated servicing spread is constant. Does the existence of non-proportional income or cost create path-dependence?

Consider the following simple transformation of the fixed dollar income (or cost):

Income per $1 of balance = Income per loan / Average loan balance (9)

Income per loan is fixed for fixed-rate loans, whereas the average loan balance gradually amortizes. The only two sources of a particular loan's amortization are the scheduled payments and curtailments (refinancing or turnover would eliminate the loan immediately). Since the scheduled amortization is market-independent at least for fixed-rate mortgages, we arrive at the following practically important conclusion: in order to apply the APD model for MSR, we have to assume that the curtailments are rate-independent.[2] This conjecture makes fixed dollar income or cost path-independent. The rest of the MSR valuation is no different from regular unstructured pass-throughs and can be carried over using pricing methods employed for the active component and the passive component, as explained above.

Residual Sources of Path-Dependence

The APD model takes care of the burnout effect, the major source of path-dependence for fixed-rate mortgages. After the decomposition is done, we

[2]Although curtailments are rate-dependent in practice, the entire curtailment speed does not exceed 1-2 CPR, for most pools. Prepayment penalties may be a counter-example.

need to review residual sources of path-dependence and arrange the numerical valuation procedure such as to reduce or eliminate potential pricing errors.

Prepayment lag, a lookback option feature, is such a source. Applications to obtain a new mortgage replacing an old one enter the origination pipeline 30-90 days before the loan is actually closed and the existing debt is paid off. Even if the prepayment model features a lag, but the backward valuation scheme is unaware of its existence, the pricing results can be somewhat inaccurate. This ignorance of the lag by the backward induction scheme usually causes small errors for pass-throughs. However, mortgage strip derivatives and MSR are highly prepayment-sensitive, and the lag may change their values in a sizable way.

It is generally known that lookbacks with fairly short lag periods can be accounted for while running the backward induction process. Let us assume, for example, that on a trinomial monthly tree speed λ_k actually depends on market rates lagging 1 month. Hence, the MBS value will also depend on both the current market and 1-month lagged market. This is to say that each valuation node of the tree should be "sliced" into 3 sub-nodes keeping track of prices matching 3 possible historical nodes 1 month back. Of course, this costs computational time; efficiency may deteriorate quickly for deeper lags and more complex trees.

Approximate alternatives do exist and it is feasible to reduce pricing errors without much trouble. AD&Co. employs a progressively sparse pentanomial tree, which does not branch every month. Branches of the tree are made from 2 to 12 months long so that the lagged market rates are explicitly known for most monthly steps. The lookback correction can also be adapted for "fractional" prepayment lag that almost always exists due to the net payment delay between the accrued-month-end and the actual cash flow date. In such a case, λ_k could be interpolated between the current-month and the previous-month values. Thus, the total lookback processing should account for both prepay lag and payment delay.

Another example of path-dependence not cured by pool decomposition is coupon reset for ARMs. Both reset caps and nonlinear relationships between the prepayment and coupon make it difficult for a backward induction scheme to account for this feature. One possible solution is to extend the state space and create an additional dimension that would keep track of the ARM coupon rate (Dorigan et al. [2001]). This state space extension will come at a cost of both computational efficiency and memory consumption. Our other study (Levin [2002]) suggests that the reset provisions found in typical ARMs allow for backward valuation with OAS accuracy within 4-5 bps, without any special measures on curing this path-dependence, under a normal term structure model. Backward valuation of ARMs remains a subject of further research.

Conclusion

The active-passive decomposition model naturally simulates the burnout and other prepayment effects, such as incentive-dependent speed ramping or "catch-up" for prepay-penalty pools. For pass-throughs and their strip derivatives, including MSRs, it splits valuation into two quick backward induction steps and produces an entire pricing grid for risk measurement at no additional cost (unlike Monte-Carlo). Whereas CMOs will still rely on Monte-Carlo as being heavily path-dependent beyond the burnout, they could benefit from better prepay modeling.

Finally, the ability to employ the backward induction pricing technique makes future values accessible, which, in itself, opens doors to new valuation and modeling tasks. We will demonstrate this interesting point in Part II of this *Quantitative Perspectives* series.

■

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QUANTITATIVE PERSPECTIVES

Introduction

In Part I of this series, we introduced an efficient valuation alternative to brute force Monte-Carlo. The Active-Passive Decomposition (APD) method splits a mortgage pass-through into two path-independent components differing in refinancing propensity, allowing for the use of the most efficient pricing methods, which operate backwards on probability trees or finite-difference grids. This valuation method runs faster than Monte-Carlo while delivering a much richer outcome — all stressed values required by mandatory risk assessments — at no additional cost. Risk managers and traders of unstructured mortgage instruments, such as agency pass-through MBS, whole loans, stripped (IO/PO) derivatives, including mortgage servicing rights (MSRs), may directly benefit from the method.

Because the model relies on the backward induction valuation performed separately for its two constituent components, it naturally assesses their future prices — very much unlike Monte-Carlo. APD sets an analytical framework where both valuation and prepayments can be modeled rigorously, beyond the traditional OAS framework and even without traditional OAS input. This type of valuation modeling is motivated by the fact that OAS, being a much better measure of expected return than yield or static spread, still falls short of explaining the dynamics of mortgage pricing. The traditional OAS measure varies across instruments (pass-throughs, CMOs, IOs, POs), their coupons, prepayment option moneyness and pool seasoning stages. Often, premium and discount MBS are priced at wider OAS than the current-coupon issues. IOs stripped off premium pools are considered hazardous, and their prices reflect mounting concerns of understated refinancing. Naturally, the respective POs look "rich," although these roles may be swapped in the discount sector where risk is associated with possible turnover overstatement. Clearly, these market phenomena defeat the very purpose of the constant OAS approach, making rich-cheap judgments inconclusive and producing shock tests that sometimes lead to very inaccurate risk measures.

The Price of Risk Constant (PORC) model developed by Bear Stearns in the 1990's (see Cohler, Feldman, and Lancaster [1997]) was, perhaps, the first approach to address these OAS shortcomings. The PORC model uses traditional time-consuming Monte-Carlo sampling, which becomes even slower due to the need of measuring values under artificially stressed prepayments. For example, every prepayment factor or uncertain parameter stressed up and down adds two new sets of Monte-Carlo runs, each possibly consisting of hundreds, if not thousands, of paths. Whereas PORC sets the same valuation goals as we do — finding prices internally adjusted for prepay risk premium - the analytical technique we employ is more efficient, even for CMOs.

For unstructured pass-throughs, we proffer a valuation model which is armed with the power of backward induction, but otherwise takes its roots from the Capital Asset Pricing Model (CAPM) and its extension, the Arbitrage Pricing Theory (APT). It allows for endogenously finding risk measures and prices for embedded prepay uncertainty in the form of return spread compensation, computed for each investment period and level of interest rates. A value obtained in this manner, therefore, will carry financially well-defined return demand and may operate without an exogenous OAS input. We will term this new method Prepayment Risk-and-Option-Adjusted Spread "prOAS" (pronounced PRO-A-S). The entire valuation model implemented on the same probability tree or finite-difference grid will deliver objective economic measures (price, OAS, duration, convexity, option cost, etc.) without any form of external pricing quotes. For example, values for IOs, POs and MSRs can be objectively derived without knowing the differences in OAS; OAS, in fact can be calculated from these economic values once they are determined. Having calibrated just two prices of prepayment risks to several widely traded MBS, we can then produce an enormous number of prices for all pass-throughs, actual or hypothetical.

We prove that, under certain assumptions, this process of explicit endogenous prepay risk assessment is mathematically equivalent to so-called "implied" prepayment modeling (see Chen [1996], Cheyette [1996], and Belbase [2001]). Such a model that we term "risk-neutral" retains the structure and features of a "physical" prepay model, but operates with risk parameters stressed to their "feared" directions (where value deteriorates). A risk-neutral prepay model easily solves the issue of CMO valuation under price of prepay risk implicitly, without having to rely on a non-feasible backward valuation. In particular, the PORC-style of handling CMOs can be refined if the explicit, time-consuming prepayment stress test is replaced by the risk-neutral prepayments.

Valuation with Prepayment Risk: Basic Concepts

For the reader's convenience, below is the pricing equation (1) from (Levin [2003]) that drives the valuation of an MBS, under the traditional OAS definition, in a single-factor arbitrage-free economy:

$$\underbrace{r+OAS}_{\text{expected return}} = \underbrace{\frac{1}{P}\frac{\partial P}{\partial t}+\frac{1}{P}(c+\lambda)-\lambda}_{\text{time return}} + \underbrace{\frac{1}{P}\frac{\partial P}{\partial x}\mu}_{\text{drift return}} + \underbrace{\frac{1}{2P}\frac{\partial^2 P}{\partial x^2}\sigma^2}_{\text{diffusion return}} \tag{1}$$

where $c(t,x)$ is the continuously paid coupon rate and $\lambda(t,x)$ is a balance amortization rate. The lone market factor x is assumed a risk-neutral diffusion with drift μ and volatility σ. We intend to consider and solve equation (1) separately for the active and the passive components of the MBS. Starting from the terminal (maturity T) condition, $P(T,x) = 1$, we operate backwards on either a finite-difference grid or on a probability tree and find pricing function $P(t,x)$ for all time steps and factor levels.

For readers who feel more comfortable with the probability tree valuation concept, we augment the PDE presentation style with explicit operations on a tree branch (see chapter 12 in Davidson et al [2003]):

$$P_{k-1} = \frac{c\tau + P_k + \lambda_{k-1}\tau(100-P_k)}{1+(r_{k-1}+OAS)\tau} \tag{1-tree}$$

where subscripts "k" and "k-1" refer to the next and previous time nodes, correspondingly, and τ is the length of the time step. Formula (1-tree) deduces the previous node price from the next node price, assuming that the tree consists of one branch. In reality, we simply weight results obtained from (1-tree) by the probability of branching.

All the terms in equation (1) represent different sources of return (see labels underneath equation), but none of them explicitly quantifies the prepayment risk. The entire compensation demand for this risk is hidden under the OAS term.

What is Prepayment Risk?

Mortgage practitioners use the term "prepayment risk" liberally. Most often they are referring to "prepayment variability," but this is not what we attempt to capture. Indeed, a large portion of prepayment uncertainty is associated with interest rates and, as such, explained by prepayment models that are inherent to modern OAS analytical systems. If prepayments were perfectly explained by a model, there would exist little reason for "prepayment risk premium." An option pricing model coupled with a "exact" prepayment formula should be able to deliver the right price for an agency-backed (default-protected) MBS operating with OAS = 0. An MBS would be valued flat to a known benchmark curve — similarly to swaptions, just with a more complex (inefficient) option exercise rule. If the MBS were not agency-backed, we would, of course, set the OAS to a relevant credit spread, but not above it. Likewise, we would add a “liquidity spread” for an illiquid MBS.

Savvy market participants realize that a model can tell only part of the prepayment story. A model's inability to predict prepayments exactly causes unexplained deviations of prepayment speeds above or below the model's forecast, often termed "prepayment surprises" or "prepayment errors." We associate the notion of prepayment risk with this uncertainty unexplained by an otherwise best-guess prepay model and assume that the OAS compensates for this risk.

Price of Risk

How would we price this risk? Suppose that the prepayment rate $\lambda(t,x,\xi)$ depends on one uncertain variable or uncertain parameter, ξ (ksi, the Greek letter x). Later in the paper, we will discuss, in detail, the exact stochastic behavioral assumptions used for this variable or parameter. For the first conceptual illustration, we assume that $\xi(t)$ is a Wiener process with zero drift and volatility of σ_ξ, i.e. $d\xi = \sigma_\xi dz_\xi$, and known initial value, $\xi(0)$. According to CAPM/APT, the risky return should be proportional to the volatility due to the risky factor ξ. A common multiplier, π_ξ, called price of risk should apply to every asset exposed to the same risk factor ξ. Using the notations above, we therefore state that for every investment period, the expected return, previously $r + OAS$ in the left-hand side of PDE (1), should be adjusted for risk as:

$$\text{Single-period expected return} = \quad r + prOAS + \underbrace{\pi_\xi \sigma_\xi \frac{1}{P}\frac{\partial P}{\partial \xi}}_{\text{prepay risk spread}} \tag{2}$$

In this expression, we introduce *prepay risk-and-option-adjusted spread* (*prOAS*) that is a "risk-free" OAS; in the absence of any other risk factors, it should be zero for a properly selected pricing benchmark. For example, we may assume that all agency MBS should be valued flat to the same-agency yield curve, on this prepay-risk-adjusted basis (*prOAS* = 0). For non-agency MBS, it should certainly account for an additional risk associated with imperfect credit and thus becomes equal to the pure credit spread that can be imposed from the S&P or Moody's rankings for a non-agency pass-through or a particular CMO tranche.

The risky spread term, unlike the traditional OAS, is not constant but rather varies with rates and age. It is also directional — can be both positive and negative — depending on the sign of price exposure to the factor ξ. Since the market provides a return premium for bearing the risk, it must make hedge instruments rich so that the properly constructed ξ-neutral portfolio of an asset and its hedge will earn a risk-free OAS, i.e. *prOAS*. This is a traditional arbitrage argument preventing constructing a risk-free portfolio that earns any excess above the risk-free return (see Hull [2000]).

Valuation Method 1: Endogenous Assessment of Risk

Factor volatility σ_ξ and price of risk constant π_ξ are defined outside the pricing model; they are common for all instruments exposed to prepay factor $\frac{\partial P}{\partial \xi}$, which we will denote as P_ξ, can be estimated internally and simultaneously in the course of backward valuation. Let us assume that we have a path-independent instrument in hand that is subject to pricing equation (1) and amortization rate λ depends on risk factor ξ.

Let us first multiply both sides of the pricing equation (1) by price P and replace the traditional expected return, $r + OAS$, with the risk-adjusted version (2):

$$(r + prOAS)P = \frac{\partial P}{\partial t} + c + \lambda - P\lambda + \frac{\partial P}{\partial x}\mu + \frac{1}{2}\frac{\partial^2 P}{\partial x^2}\sigma^2 - \underbrace{\pi_\xi \sigma_\xi P_\xi}_{\text{price of risk}} \tag{3}$$

Second, we take the first derivative of both sides of equation (3) with respect to ξ:

$$(r + prOAS)P_\xi = \frac{\partial P_\xi}{\partial t} - P_\xi\lambda + \frac{\partial P_\xi}{\partial x}\mu + \frac{1}{2}\frac{\partial^2 P_\xi}{\partial x^2}\sigma^2 + (1-P)\frac{\partial \lambda}{\partial \xi} - \underbrace{\pi_\xi \sigma_\xi P_{\xi\xi}}_{\text{term to omit}} \tag{4}$$

where, for now, we intend to disregard the last term. We have obtained a system of linear partial differential equations (3), (4) containing two unknown functions, $P(t,x)$ and $P_\xi(t,x)$. It can be simultaneously solved backwards on a finite difference grid or a probability tree starting from the terminal (maturity T) conditions: $P(T,x) = 1, P_\xi(T,x) = 0$. This backward induction is carried out on a usual (t,x)-grid or probability tree for one value of $\xi = \xi(0)$ and separately for the active and passive part. To clarify, we augment the presentation with a single-branch tree analog of system (3), (4) with the $P_{\xi\xi}$ term omitted:

$$P_{k-1} = \frac{c\tau + P_k + \lambda_{k-1}\tau(100 - P_k) - (P_\xi)_k\tau}{1 + (r_{k-1} + prOAS)\tau} \tag{3-tree}$$

$$(P_\xi)_{k-1} = \frac{(P_\xi)_k(1 - \lambda_{k-1}\tau) + (\lambda_\xi)_{k-1}\tau(100 - P_k)}{1 + (r_{k-1} + prOAS)\tau} \tag{4-tree}$$

where $\lambda_\xi = \frac{\partial \lambda}{\partial \xi}$, a measure known from the definition of prepayment factor ξ. Once again, formulas (3-tree) and (4-tree) will have to incorporate probability weighting for the actual tree.

Whether we use the prOAS method or the OAS method, we generate cash flows for every node of the main pricing grid only once. Unlike the PORC approach, we never re-generate them for stressed values of ξ because we know, analytically, how amortization speed (λ) depends on the risk factor (or factors) and how it enters the pricing model. It is important to emphasize that we do not intend to expand our pricing grid to the ξ-dimension because we do not look to explore the full price dependence on ξ; we target only one additional measure, P_ξ, needed for our risk-adjusted valuation scheme.

Our seemingly frivolous omission of the second derivative term that makes equation (4) approximate is motivated by a desire to maintain a closed mathematical construct, i.e. to match the number of unknown functions with the number of equations. Had we included the second derivative term, $P_{\xi\xi}$, we would have needed another equation for it, which can't be derived without the third derivative, etc. An attempt to compute all derivatives exactly will force us to create a special ξ-dimension for our pricing grid — much in the same way as we normally do for the x variable. Since such a full-scaled approach would considerably deteriorate computational speed and efficiency, we prefer to drop the "redundant" higher ξ-derivatives from the system. Mathematically, we account for a full dependence on ξ in the traditional PDE (1), include price of risk, but ignore its own ξ-dependence. In all practicality, our simple approach will capture this risk approximately because prepayment duration is measured without considering prepayment convexity. In the next section, we discuss adding the prepay convexity term to the pricing equation.

Adding Prepayment Convexity Cost

Since the price of an MBS is typically nonlinear in prepayment speed, we should generally assume it is nonlinear in factor ξ. Since this is a volatile factor, a convex or concave relationship may serve as a source of additional return, positive or negative. Levin and Daras [1998] show that the convexity term is positive for premium pass-throughs and negative for discounts; this additional source of expected return or loss grows with the premium or discount, easily reaching the respectful range of 5 - 10 basis points of OAS typically hidden from the market trades. For our Wiener process $\xi(t)$, equation (3) should include the $1/2 P_{\xi\xi}\sigma^2_{\xi}$ diffusion term, in its right-hand side. This term mathematically represents the expected return measured over an infinitesimal horizon and then annualized; with this addition, PDE (3) is extended to:

$$(r + prOAS)P = \frac{\partial P}{\partial t} + c + \lambda - P\lambda + \frac{\partial P}{\partial x}\mu + \frac{1}{2}\frac{\partial^2 P}{\partial x^2}\sigma^2 - \underbrace{\pi_{\xi}\sigma_{\xi}P_{\xi}}_{\text{price of risk}} + \underbrace{\frac{1}{2}P_{\xi\xi}\sigma_{\xi}^2}_{\text{prepay convexity cost}} \tag{5}$$

It is important to note that this convexity cost is virtually unrelated to the price of risk and is not governed by π_{ξ}. In fact, we could account for this term under the traditional OAS approach, just assuming an additional pricing factor, $\xi(t)$. Price of risk and the convexity cost are two principally different sources of return. "Risky" return is artificially rewarded by the market for bearing the risk, whereas convexity cost is a certain (systematic) return realized in a volatile market. Since equation (5) now deals with two unknown derivative functions, P_ξ and $P_{\xi\xi}$, we will need to add more equations to close this mathematical construct. We differentiate PDE (5) twice, each time omitting the third derivative with respect to ξ:

$$(r + prOAS)P_\xi \approx \frac{\partial P_\xi}{\partial t} - P_\xi \lambda + \frac{\partial P_\xi}{\partial x}\mu + \frac{1}{2}\frac{\partial^2 P_\xi}{\partial x^2}\sigma^2 + (1-P)\frac{\partial \lambda}{\partial \xi} - \pi_\xi \sigma_\xi P_{\xi\xi} \tag{6}$$

$$(r + prOAS)P_{\xi\xi} \approx \frac{\partial P_{\xi\xi}}{\partial t} - P_{\xi\xi} \lambda + \frac{\partial P_{\xi\xi}}{\partial x}\mu + \frac{1}{2}\frac{\partial^2 P_{\xi\xi}}{\partial x^2}\sigma^2 + (1-P)\frac{\partial^2 \lambda}{\partial \xi^2} - 2P_\xi \frac{\partial \lambda}{\partial \xi} \tag{7}$$

System (5) - (7) now contains three partial differential equations for three unknown functions of time and market: $P(t,x)$, $P_\xi(t,x)$, and $P_{\xi\xi}(t,x)$. They are solved backwards and simultaneously, starting from the terminal conditions, $P(T,x) = 1$, $P_\xi(T,x) = P_{\xi\xi}(T,x) = 0$. Using the notions of contemporary finance we could classify models (3) - (4) or (5) - (7) as equilibrium pricing models with explicit prepay risk accounting.

Valuation Method 2: Risk-Neutral Prepayment Modeling

The Concept of a Risk-Neutral Prepayment Model

Pricing equation (3) allows for an important financial interpretation. Suppose we still work with the traditional PDE (1) (i.e., ignore the price of risk), but now let the risk factor ξ drift with a negative $\pi_\xi \sigma_\xi$ rate per year:

$$d\xi = \sigma_\xi dz_\xi - \pi_\xi \sigma_\xi dt \tag{8}$$

It is easy to see that such a drift contributes a systematic return that is mathematically identical to the above marked "prepay risk spread". Indeed, the full time derivative term, $\partial P/\partial t$ in the right-hand side of equation (1) will now be comprised of a $\partial P/\partial t$ term measured due to a simple passage of time (i.e. with unchanged ξ) minus a $\pi_\xi \sigma_\xi P_\xi$ term that comes from the Ito Lemma applied to price P as a function of random variable ξ defined by process (8). Which directly leads to the pricing equation (3) with risk.

Those who build, use or simply understand the concept of arbitrage-free modeling may find themselves in familiar waters. Indeed, pricing with risk can be replaced with pricing without it, but with the risk factor ξ drifting in the feared direction. The rate of this drift is proportional to volatility σ_ξ, with the coefficient of this proportionality being the price of risk, π_ξ. This is the same concept used when constructing term structure models and pricing financial derivatives while taking the forward rates and prices into consideration.

A prepayment model with its factor ξ set to drift at the risk-adjusted rate can be logically called a *risk-neutral prepayment model* — as is the case with all other financial models that explore this concept. It is therefore meaningless to wonder how well such a model fits actual prepayments - it is simply not meant to do that job. There is also no point in arguing that the risk-neutral drift for factor ξ may be partly caused by a model's systematic bias. For example, the market may expect the refinancing process to be more efficient going forward than it has been. Mathematical consequence for this bias cannot be distinguished from the price of risk: it results in a drift change for ξ. Using the risk-neutral prepay model and a surely known prOAS parameter should lead to the same valuation results as using the traditional OAS approach coupled with the "actual" prepayment model.

Mortgage market participants are accustomed to the fact that brokers and independent analysts differ widely when forecasting prepayments and reporting OAS numbers for the same instruments and market conditions. The transition from an actual model, which usually employs historical prepayments, to a risk-neutral one that targets known prices for known mortgage instruments compensates for the disparities. Hence, judgements regarding risk-neutral prepay speeds should vary less once bound to the same set of prices and a prOAS benchmark across the market. From the times of Black and Scholes, risk-neutral modeling has been known for its objectivity, leaving less bias for systematic, model-infused errors.

Although we've shown that the concept of a risk-neutral prepay model is theoretically equivalent to the explicit risk assessment performed for each of the investment periods, from a financial engineer's point of view, a fundamental difference exists between these two techniques. The explicit risk assessment method computes a partial derivative P_ξ directly for each investment period and rate level. This task is feasible if the entire valuation is performed backwards and the MBS price and its derivatives, with respect to risky factor x, can be found for every node of the pricing grid. For example, unstructured MBS can be priced this way using the "burnout-curing" APD idea. CMOs, on the other hand, are heavily path-dependent beyond the burnout and are therefore not subject to backward valuation. For CMOs, the risk-neutral prepay model would be an ideal method, as it does not require computing P_ξ directly. Letting the risk factor ξ drift in the feared direction naturally explores the price's dependence on ξ without measuring P_ξ explicitly.

Risk-Neutral Prepayment Model with Backward Induction

A risk-neutral prepay model retains the structure of the actual model: if the latter was built in the APD form, so will the former. It means that one can exploit the backward induction technique with one notable change: set price of risk π_ξ to zero in the PDEs because the risk adjustment is now fully embedded in the prepay model. With this minor modification, system (5) - (7) can be used solely to account for the prepayment convexity return component.

Arguably, this approach is even more accurate than solving system (3) - (4) or (5) - (7) with the actual prepay model and non-zero price of risk π_ξ. As we pointed out, the above listed PDEs are only approximate ones with higher derivative terms omitted. For example, even if we decide to exclude the prepay convexity return from the very beginning, we still can only compute the required derivative P_ξ approximately. Under the same assumption, equation (1) remains exact if we replace OAS with prOAS and the actual prepay model with the risk-neutral one.

Once again, for the proposed backward valuation method, we do not "simulate" random factor $\xi(t)$, nor do we construct a special grid for possible values of ξ. All we do is allow the risk factor $\xi(t)$ to now follow the pure drift process, $d\xi = -\pi_\xi \sigma_\xi dt$ or, equivalently, $\xi(t) = \xi(0) - \pi_\xi \sigma_\xi t$ instead of $\xi(t) = \xi(0)$. The random term is not simulated, but kept quantified (albeit approximately) in the model (5) - (7) through prepayment convexity.

Risk-Neutral Prepayment Model in Monte-Carlo

If we are forced to employ Monte-Carlo (for example, in the case of

CMOs), we simply replace a physical prepay model with the risk-neutral one. If we wish to account for prepay convexity, then we should augment the risk-neutral prepay model with one extra step: randomly simulate $\xi(t)$ according to stochastic equation (8) when sampling regular Monte-Carlo paths. This is not a burden at all; the number of random factors affects neither the theoretical convergence nor the accuracy of Monte-Carlo. Therefore, we can randomize the market factor $x(t)$ and the prepay risk factor $\xi(t)$ simultaneously, according to their respective stochastic models. Of course, no additional risk adjustment has to be made to present valuing random cash flows.

Chart 1

Summary of OAS and prOAS Valuation Methods

	Regular OAS Valuation	prOAS Valuation without Prepayment Convexity	prOAS Valuation with Prepayment Convexity
Unstructured pass-through with APD prepay model*)	Solving equation (1) backwards	Solving systems or equations (3), (4) backwards, **OR** Solving equation (1) backwards with *OAS* = *prOAS* and the risk-neutral prepay model	Solving systems or equations (5) - (7) backwards, **OR** Solving equations (5) - (7) backwards with $\pi_{\xi} = 0$ and the risk-neutral prepay model
CMOs and other MBS	Regular Monte-Carlo	Regular Monte-Carlo operating with the risk-neutral prepay model	Monte-Carlo operating with the risk-neutral prepay model and randomized prepay risk factor

*) For solving PDEs, apply backward induction separately for the active and passive components of mortgage pool

Properties of Prepay Risk Factors

In the above conceptual presentation of the prOAS model, we simplified for the sake of clarity. For example, we assumed that the actual prepay risk factor is a simple Wiener process (Brownian motion). Some random variables can often be fairly modeled as arithmetic or geometric Brownian motions in financial applications, but not all. In particular, errors produced by prepay models tend to grow as the time horizon extends at a much slower pace than the $\sqrt{t}$ rule suggests. For instance, if a prepay multiple was a Wiener process, the prepayment uncertainty would be 3.5 times larger for the 1-year horizon than for the 1-month horizon. A well-observed fact about prepay surprises in an unbiased model is that they can be fairly large, even in a month, but will not grow unboundedly over years. Such a stochastic pattern suggests that the pure Brownian motion pattern we started with may be an inappropriate assumption therefore forcing us to consider some alternatives.

Mean-Reverting Risk Factor Pattern

Instead of assuming that actual factor $\xi(t)$ is a pure Brownian motion i.e.,

$d\xi = \sigma_\xi d_\xi$, we can consider a mean-reverting model:

$$d\xi = a_\xi(\bar{\xi} - \xi)dt + \sigma_\xi dz_\xi\ ,\ \xi(0) = \bar{\xi} \qquad (9)$$

where a_ξ is the mean reversion parameter, and $\bar{\xi}$ is the long-term equilibrium.

How would a mean-reverting prepayment risk change the prOAS valuation model? First, the existence of a drift term in (9) leads to the appearance of one in the right-hand sides of PDEs (3) and (5); this extra term will be $P_\xi a_\xi(\bar{\xi} - \xi)$. Second, since we assess the cash flows only for the unbiased value of ξ, this valued at $\xi = \bar{\xi}$, will momentarily turn to zero in (3) or (5). However, in transitions to equations (4), (6), and (7), this term produces non-zero derivatives. For example, we should add a $-2P_\xi a_\xi$ term to the right-hand sides of equations (4) and (6), and a $-2P_{\xi\xi} a_\xi$ term to the right-hand side of equation (7). Therefore, it is easy to modify the prOAS method for mean-reverting risk factors. Equivalent volatility parameter, σ_ξ, should now be somewhat larger than if we assumed a Wiener process pattern — mean reversion suppresses future deviations.

If we decide to operate with a risk-neutral prepayment model, we would add a $-\pi_\xi\sigma_\xi$ drift rate, thereby making the dynamics of our risk factor subject to a risk-neutral version of equation (9):

$$d\xi = [a_\xi(\bar{\xi} - \xi) - \pi_\xi\sigma_\xi]dt + \sigma_\xi dz_\xi\ ,\ \xi(0) = \bar{\xi} \qquad \text{(9-rn)}$$

Its solution without the random term is

$$\xi(t) = \bar{\xi}e^{-a_\xi t} + (1 - e^{-a_\xi t})(\bar{\xi} - \frac{\pi_\xi\sigma_\xi}{a_\xi}) \qquad (10)$$

These are the dynamics of the risk factor $\xi(t)$ when using a risk-neutral prepayment model. For the "physical" prepayment model, $\xi(t)$ starts from $\bar{\xi}$ and simply remains unchanged. Risk neutrality does not alter the risk factor's starting point, but lowers its long-term equilibrium by $\pi_\xi\sigma_\xi / a_\xi$. We can employ dynamics (10) for $\xi(t)$ when "inserting" a risk-neutral prepayment model in lieu of a physical model. This includes all variations of the backward induction method (with and without prepayment convexity terms) and the case of Monte-Carlo pricing without prepayment convexity. For the use of Monte-Carlo pricing with prepayment convexity, we randomize $\xi(t)$ as dictated by the stochastic differential equation (9-rn).

Factors as Uncertain Parameters (Single-Jump Risk Pattern)

Instead of treating risk factors as variables ("processes"), we can conceive them as uncertain parameters. This means that our risk factor ξ will not change over time — it is an unknown constant drawn once from some distribution. In fact, this interpretation of prepayment risk factors underpins the PORC model, and it can be derived as a special case of our general prOAS model.

Let us assume that volatility σ_ξ is time-dependent and not constant, namely, its square is proportional to the Dirac function, $[\sigma_\xi(t)]^2 \sim \delta(0)$. This means that volatility is zero for any $t > 0$ and infinite when $t = 0$. The integral of $[\sigma_\xi(t)]^2$ over any period of time is finite, constant, and equal to the variance of random parameter ξ; let us denote it $[\sigma_{\xi\,jump}]^2$. When performing the backward induction, we will find that both risk and prepay convexity terms contribute no return until we reach the very last time step, $t = 0$. Of course, this last-period return will suddenly become infinite. An infinite return, realized over an infinitesimal instance of time is equivalent to an immediate price shock. This price shock is comprised of two terms: the "risk" shock (negative $\pi_\xi \sigma_{\xi\,jump} P_\xi$) and "convexity" shock $[1/2 P_{\xi\xi}(\sigma_{\xi\,jump})^2]$. Hence, our backward induction process will keep assessing derivatives P_ξ and $P_{\xi\xi}$, only to adjust price at time $t = 0$.

If we decide to apply a risk-neutral prepay model, we would simply reduce the actual value of ξ by $\pi_\xi \sigma_{\xi\,jump}$. Therefore, a risk-neutral prepay model with a random risky parameter is just a model with this parameter stressed in the feared direction. At the end of this modified backward valuation, we adjust the price by a prepay convexity term. Finally, if the Monte-Carlo is elected as the valuation technique, we will randomize parameter ξ along with the main market factor $x(t)$.

Combined Single-Jump-Diffusion Behavior

A single stochastic pattern may fall short of explaining prepayment risk. For example, a diffusive pattern would only be valid if the prepayment model perfectly forecasted the near-term speeds. However, even a newly designed prepayment model, validated over historical intervals, is not free from the risk it is already biased. Conversely, a single-jump pattern that reduces risk to a single, time-independent parameter is virtually unseen in other applications of quantitative finance: financial risks always grow as the time horizon extends.

Considering a combined, single-jump-diffusion pattern may be advantageous. It allows for existence of both types of risks, one for time zero and another mounting forward. When we employ the backward induction, we should account for the diffusive risk first by adding (or subtracting) the risk spread for each investment period — as we normally do. At the end of the valuation (i.e. at time zero) we should correct the obtained price for the single-jump risk.

If we prefer to use a risk-neutral prepayment model with the single-jump-diffusion set-up, we stress the initial condition for factor ξ down by $\pi_\xi \sigma_{\xi\,jump}$ (in recognition of the single-jump risk) and let $\xi(t)$ exponentially decline further by $\pi_\xi \sigma_\xi / a_\xi$ as determined by the mean-reverting diffusive risk (see Part III for an illustrative pattern).

Calibration Issues

In Part III, we will present a practical calibration exercise in which we discuss a two-risk-factor model. It is clear that the parameters of the model, price of risk π_ξ, factor volatility σ_ξ and mean reversion a_ξ need to be established before we put a prOAS model to work.

Stochastic properties of factor ξ can be measured when the prepayment model is built. If ξ reflects prepayment errors, it can be viewed as the residual process, with all needed properties (variances and auto-covariances) revealed. In the absence of such evidence, MBS experts can rely on subjective intuition. Indeed, no matter how σ_ξ and a_ξ are defined, it is essential that we find the price of risk, π_ξ; this choice, bound to actual market prices, may reduce errors introduced when using improper σ_ξ and a_ξ.

TBA MBS are the most liquid mortgage instruments to use for the calibration set. Starting with a prOAS target set to the relevant agency-swap spread (say, -10 bps), we would like to tune the price of risk so as to minimize the pricing errors (i.e. the difference between actually quoted prices and values derived from our prOAS model). As we will show in Part

III, prepayment risk is not single-dimensional, and risk factor ξ and all its attributes should actually be understood as a vector of two factors, refinancing and turnover. This interpretation drastically improves TBA pricing fit under the prOAS model, explaining why both premium and discount instruments can be priced at a somewhat elevated OAS.

Agency IOs and POs are two other popular instruments that range considerably in traditional OAS quotes — from a negative few hundred basis points to a positive thousand and more. In Part III of this series, we show that this range is generally in line with fundamentals of the prOAS method. The risk adjustments seen in the TBA pricing are drastically magnified in the strips. It is easy to see when using the risk-neutral prepayment model as an example. These models, as we previewed, will likely feature higher refinancing and lower turnover than the corresponding "physical" prepay models. Such a transition will affect IOs and POs to a much larger extent than it did the TBAs. Nevertheless as we will show in Part III, pricing of strips is not always consistent with TBAs, leaving room for riskless arbitrage.

Since MBS strips' prices carry relatively large risk components, they may reasonably be used directly for calibrating the prices of risk and other prOAS parameters. Considering a set of (IO, PO) pairs stripped from the same collateral, we can even slightly enhance the calibration process. Instead of letting the prOAS target be known a priori, as we did for TBAs, we can constrain the calibration process so as to achieve the same prOAS for IO and PO — the result may not be identical to the observed agency debt spread.

Using strips for calibration, as attractive as it sounds, has a few known drawbacks: limited liquidity and substantial market technicality. An analyst must apply care when selecting a benchmark set, since many agency IOs are owned by a single investor and their prices are not always indicative of the value.

Conclusion

We have presented a dual-use theoretical framework for the valuation of MBS adjusted for both prepayment option and prepayment uncertainty.

First, we proposed an explicit risk computation method that relied on the backward induction technique, which operates separately on the mortgage pool's active and passive constituents. For every investment period and every interest rate level, our analytics add (or subtract) periodic spread based on internally assessed prepayment risk.

This method, feasible only for active-passive-decomposed pass-throughs, has an equivalent mathematical alternative we call risk-neutral prepayment model. With a prepayment factor drifting to its feared direction, we are able to implement the prepay-risk-and-option-adjusted valuation with the Monte-Carlo sampling that is mandatory for CMOs.

In either case, instead of relying on a traditional (often uncertain) OAS input, this method suggests pricing all agency MBS, including IOs, POs, and MSRs, flat to agency debentures.

In the final part of this QP trilogy, we will put the prOAS model to work. We will specify the abstract risk factor ξ as a two-dimensional vector with refinancing and turnover prepayment multiples being the actual prepayment risk factors. We will show how the model works for TBAs, IOs and POs. We will provide a unique view of an MBS "risk map" clearly indicating demanded spread compensation by age and rate level. We will also assess how conventional rate risk measures (duration, convexity) change when transitioning from a constant-OAS model to our new prOAS approach.

Acknowledgements

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PART III: A prOAS VALUATION MODEL WITH REFINANCING & TURNOVER RISK

JUNE 2004

by Alex Levin

Introduction

In this third part of our "Divide & Conquer: Exploring New OAS Horizons" series on new methods in MBS valuation, we develop practical applications using the theory established in the previous two parts. The theoretical framework utilizes an active-passive decomposition model (Levin [Sept. 2003]) and the concept of prepay risk-and-option-adjusted valuation (Levin [Mar. 2004]). These publications are summarized below, but readers may want to review them before reading this one.

In Part I, we decomposed mortgage collateral into two pieces differing in their refinancing propensity. Each piece can additively combine various sources of prepayments such as refinancing, turnover, etc. Each source is constructed to be path-independent. The pool, as a whole, burns out and remains path-dependent as its composition changes over time, but its active and passive constituents are path-independent and can be quickly priced backwards, starting at maturity.

In Part II, we proposed a valuation approach that is "beyond OAS" and essentially comes from the capital asset pricing model (CAPM) and its extension, the arbitrage pricing theory (APT). Having attributed the very existence of a positive OAS to prepayment risk, we quantified OAS as the sum of "risk-free" OAS, which we called ***prOAS*** (prepayment-risk-and-option-adjusted spread), and an explicitly measured risk spread compensation received by investors and paid by hedgers. This risky spread term is found by solving modified pricing equations, again using the APD split and backward induction.

We have also proven that this risk-adjusted valuation method is mathematically equivalent to using a traditional OAS approach with a "risk-neutral" prepayment model instead of a "physical" one. Such a model features the risk factor (we called it ξ) drifting in the feared direction with a rate of $-\pi_\xi \sigma_\xi$, where σ_ξ is the factor's volatility and π_ξ is the market price of risk; both are common for all MBS exposed to ξ. This important finding allows the use of Monte-Carlo simulations to implement the prOAS idea; from there we extend the prOAS method to CMOs.
We posit that, on a prOAS basis, liquid agency MBS should be priced flat to agency debentures. This makes the prOAS input known with a much higher precision than traditional OAS, which varies widely across instruments (pass-throughs, CMOs, IOs, POs), their coupons, prepayment option moneyness and pool ages.

In order to carry out the prOAS valuation using the APD modeling framework, we must first interpret what has been until this point an abstract risk factor, ξ. This concluding part of the series will guide readers through practical modeling and valuation exercises. We will introduce a two-risk-factor model, which includes refinancing and turnover scales as independent sources of the prepay risk. After giving this model proper description and specification, we will illustrate how it can be calibrated to the TBA market using so-called "principal components" of the OAS. We will discover interesting and important phenomena in the MBS market. First, we will observe how periods of relative stability in prices of risk may be interrupted by their drastic changes in times of panic. Second, we will prove the existence of a systematic price bias in Trust IOs that seems to be driven solely by the fear of fast refinancing and completely misses their hedging power against the turnover risk.

A Two-Risk-Factor Model

We briefly mentioned in Part II that both premium and discount MBS are often traded at somewhat elevated OAS. IOs stripped from premium collateral are considered hazardous and priced progressively cheaper (on an OAS basis) than IOs taken from discount or current-coupon collateral. This observation leads to the following conjecture: two major prepayment sources, the refinancing process (for the premiums) and the turnover process (vital for understanding the discounts), are perceived to be risky by the mortgage market. To put it into simple practical terms, there exist two distinct market fears — refinancing understatement and turnover overstatement. Hence, the model of risk should be two-dimensional, at least, which is a rather simple extension of our general theoretical set-up in Part II. Allowing the refinancing and the turnover processes to be randomly scaled, we can extend the simple APD model (Part I) as follows:

$$\begin{aligned} \text{ActiveSMM} &= \rho * \text{RefiSMM} + \tau * \text{TurnoverSMM} \\ \text{PassiveSMM} &= \rho * \beta * \text{RefiSMM} + \tau * \text{TurnoverSMM} \end{aligned} \qquad (1)$$

where prepay multipliers ρ and τ are uncertain, but centered on 1. Thus, instead of one hypothetical prepay risk factor ξ, we now have two, ρ and τ. Every risk premium and convexity cost found in the pricing equations in Part II now becomes a simple sum of two, associated with the refinancing risk and the turnover risk, correspondingly. In doing so, we assume known volatilities, σ_ρ and σ_τ of two Wiener processes, $\rho(t)$ and $\tau(t)$, as well as two prices of risk, π_ρ and π_τ. Further, due to the model's specification (1), partial derivatives of total amortization speed (λ) taking with respect to ρ and τ are:

$$\frac{\partial \lambda}{\partial \rho} = \text{RefiSMM for the active part, or } \beta * \text{RefiSMM, for the passive part;}$$

$$\frac{\partial \lambda}{\partial \tau} = \text{TurnoverSMM for either part,}$$

and the second derivatives are all zeros since the SMMs are linear in risk multipliers.

The conceptual risk-adjusted return formula (2) from Part II now becomes

$$\text{Single-period expected return} = r + prOAS + \underbrace{\pi_\tau \sigma_\tau \frac{P_\tau}{P} - \pi_\rho \sigma_\rho \frac{P_\rho}{P}}_{\text{prepay risk spread}} \qquad (2)$$

where P_τ and P_ρ stand for partial derivatives: $P_\tau \equiv \dfrac{\partial P}{\partial \tau}$, $P_\rho \equiv \dfrac{\partial P}{\partial \rho}$.

Note the negative sign in front of the refinancing risk. Since premium fixed-rate MBS typically have $P_\rho < 0$, we can reward them by either assuming negative price of risk constant, π_ρ, or using the negative sign in the spread formula. Discount fixed-rate MBS have $P_\tau > 0$, so the positive sign in (2) produces positive return compensation for bearing the turnover risk.

Extending our illustrative, single-risk-factor theory from Part II is rather simple. As before, we start with the OAS pricing partial differential equation (PDE), shown in both Part I and Part II,

$$\underbrace{r+OAS}_{\text{expected return}} = \underbrace{\frac{1}{P}\frac{\partial P}{\partial t}+\frac{1}{P}(c+\lambda)-\lambda}_{\text{time return}} + \underbrace{\frac{1}{P}\frac{\partial P}{\partial x}\mu}_{\text{drift return}} + \underbrace{\frac{1}{2P}\frac{\partial^2 P}{\partial x^2}\sigma^2}_{\text{diffusion return}} \qquad (3)$$

We then replace the expected return, $r + OAS$, with expression (2). Once modified, risk-adjusted PDE will include unknown derivatives P_τ and P_ρ. The next step is to differentiate this equation with respect to each risk factor, ρ and τ, thereby adding two more equations and completing the mathematical construct. The total number of pricing PDEs to solve will be either three with prepay convexity cost disregarded (for P, P_ρ and P_τ) or six with prepay convexity cost included (for P, P_τ, P_ρ, the second derivatives $P_{\rho\rho}$, $P_{\tau\tau}$ and the mixed derivative[1] $P_{\rho\tau}$). The valuation process under the two-risk-factor prOAS model will certainly run more slowly than for constant OAS pricing. However, the total computational time does not grow three or six times since many operations are identical (cash flow generation, to mention the most time-consuming one). In the AD&Co. system, valuation under the full-scale two-risk-factor prOAS model is about half as fast as the one with constant OAS. Nevertheless, it remains a good deal faster than Monte-Carlo, while offering a much richer outcome. Its fundamental relationship to the CAPM/APT provides for a sound theoretical foundation, and implementation on a grid reveals well-defined spread compensation by age and rate level — information which can not be delivered by other known models.

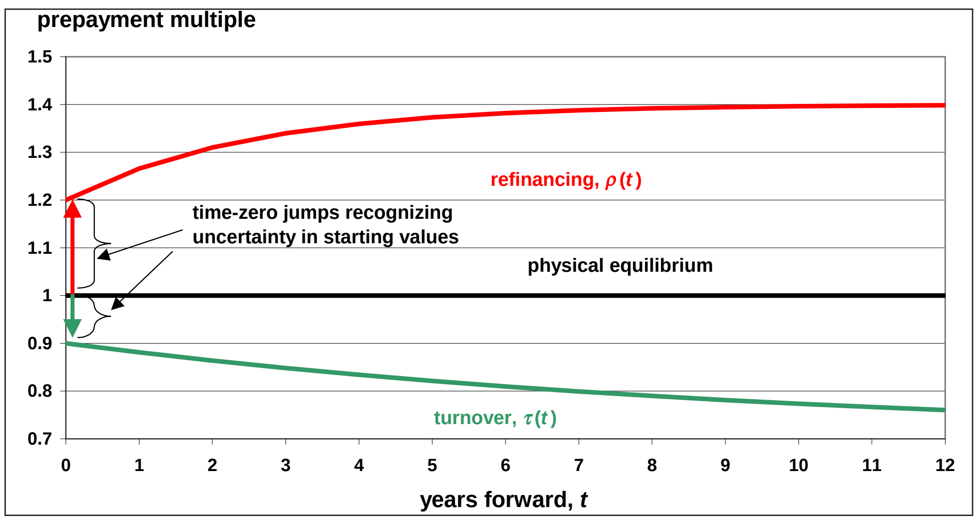
For this two-factor prepay risk setting, constructing a risk-neutral prepay model remains an equivalent alternative method of computing prOAS: we accelerate refinancing (set ρ to drift above 1 at the rate of $\pi_\rho\sigma_\rho$) and decelerate turnover (set τ drifting below 1 at the rate of $\pi_\tau\sigma_\tau$). All claims made in Part II about the relationship between these two valuation ideas apply entirely to the two-factor risk model. The risk-neutral prepayment model can be used for both backward valuation and forward sampling. If it is used in conjunction with backward valuation, then constants π_ρ and π_τ are set to zero in pricing PDEs. In particular, this substitution eliminates the mixed derivative term $P_{\rho\tau}$. If it is used for forward sampling (within

[1]The mixed derivative term appears in the model even if risk factors are assumed independent.

a Monte-Carlo valuation framework), then factors $\rho(t)$ and $\tau(t)$ are set to deterministically drift to their feared directions at their respective risk-neutral drift rates (without randomization). If it is required to compute prepayment *convexity* cost in addition to prepayment *uncertainty* cost, then risk factors $\rho(t)$ and $\tau(t)$ are randomized following their respective, risk-neutral stochastic models.

For a set-up that combines single jump and mean-reverting diffusion (also considered in Part II), these risk-neutral drifts for the refinancing and the turnover multiples are shown in Figure 1. We can virtually "split" the risk between jump and diffusion and alter the rate of reversion. The dynamics shown in Figure 1 associate about half of refinancing risk with refinancing uncertainty already existing at time zero; the other half appears gradually. We are generally more certain about the starting turnover rate, and it may take some time before future macroeconomic conditions will alter it; so a smaller portion of the total risk-neutral move is present at time zero for the turnover risk component.

Figure 1
Risk-neutral Prepay Multiples



Note: This figure is for illustration only and does not constitute an actual AD&Co. risk-neutral model

Putting the Model to Work

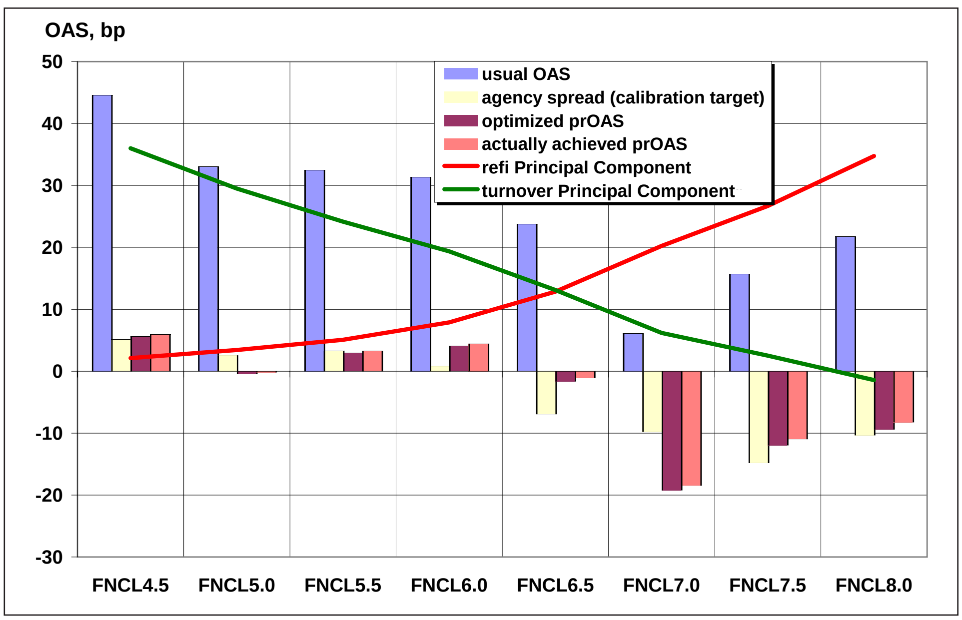
At this point, we will steer the theoretical construct of this paper toward practical steps for using the prOAS model. We will also discuss problems prOAS may address as well as valuation results it generates.

Calibration to TBAs

Assuming we know volatility and mean reversion parameters for the risk factors $\rho(t)$ and $\tau(t)$, we need to assign ("calibrate") price of risk constants, π_ρ and π_τ. We can tune these constants to match market prices (or the OAS) for a range of actively traded securities. To obtain the results presented in Figure 2, we worked with eight TBA instruments priced on August 29, 2003 with net coupons ranging from 4.5% to 8.0%. On that day the mortgage current coupon rate was 5.67%, thus there were premiums and discounts in our sample. First, we measured OAS numbers using the traditional valuation concept, without any risk adjustment (blue bars). We then employed the prOAS pricing method and selected price of risk constants π_ρ and π_τ, so as to bring the prOAS levels (purple and orange bars) as close as possible to the agency spreads to swaps (yellow bars). Fannie Mae debentures carry a full government guarantee and no prepayment risk; their rates may serve as a well-defined target for the prOAS measure.[2] As seen, we managed fairly well across the range of TBAs, having achieved a 4 basis point root-mean-squared accuracy in reaching the debenture prOAS target.

Figure 2

Prices of Risk Calibration to TBAs as of 08/29/03



The lines drawn in Figure 2 show the "principal components" of OAS. The red line measures traditional OAS addition due to the refinancing risk. The green line shows the same for the turnover risk. Directionalities for both lines are apparent, but some interesting points are notable. For example, the green line almost never leaves positive territory. Discounts would certainly lose value with slow turnover, but why will premiums

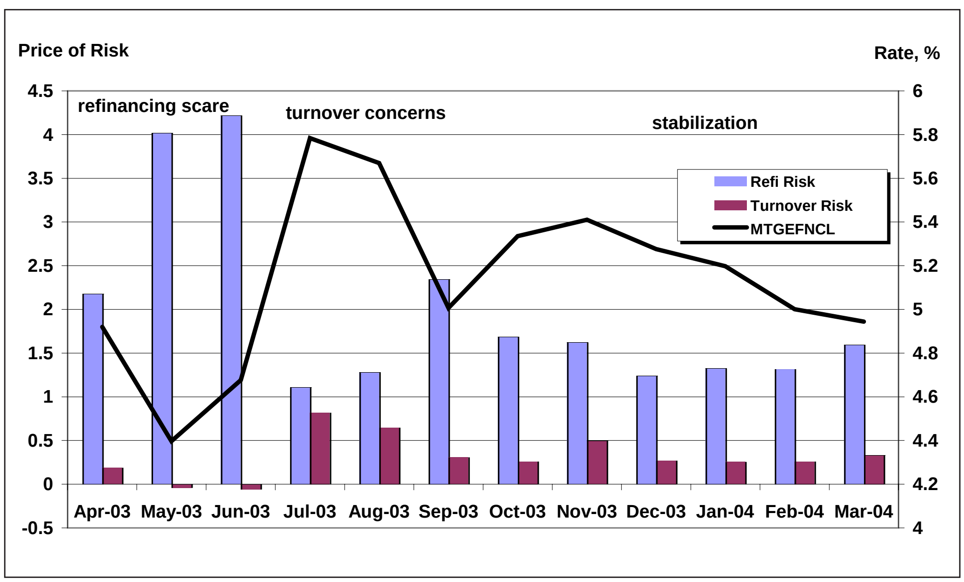
[2] Arguably, TBAs should even trade slightly rich to the agency debt curve because they (a) have superior liquidity and (b) are collateralized. We leave these points for future consideration.

suffer? The very steep yield curve is primarily responsible for this effect: slowing turnover pushes cash flows to longer maturities with higher discount rates. Slower turnover also somewhat increases time value of the prepayment option. Optimization using principal components is approximate, so actually achieved prOAS levels are sub-optimal (orange bars).

Dynamics of Risk

Will parameters of the prOAS model exhibit stability over time, or do we need to calibrate the model on a daily basis? While a goal of the physical models is to stay steady, the concept of risk-neutrality is bound to changing market prices for benchmark instruments, which reflect the dynamics of market preferences for risk. Since the prOAS model is built on the idea of risk-neutrality, this model will require a live market feed. If the market prices for TBAs exhibit visible OAS tightening and widening over time, then they are sending us a message of changing perception of prepayment risk. This conjecture is borne by examining trends in results of the calibration of the prices of risk constants π_ρ and π_τ at different dates, as shown in Figure 3. These parameters are not constant and even demonstrate an exaggerated reaction to interest rate dynamics.

Figure 3
Dynamics of Prepayment Risk



When rates dropped to their 40-yr record lows (May-June 2003), refinancing fears reached the stage of panic. High premiums (FNCL7.5 and FNCL8.0) did not appreciate, which meant that their Libor OAS levels increased to 100 bps and above to absorb much of the rate plunge. During that period, the calibrated price of refinancing risk surged. No concern about turnover was in the calibration, as the discount sector evaporated.

When rates moved back up through summer of 2003, the refinancing wave started to cool off; large volumes of freshly originated FNCL4.5 and FNCL5.0 became discounts. This was the period when the turnover concerns became apparent. The rest of the time, we witnessed a general stabilization of the risk prices.

Comparing the heights of purple and blue bars in Figure 3, one may draw the conclusion that the MBS market is systematically dominated by the refinancing, not the turnover fears. This is a possible misperception — as shown in Figure 2, the principal components of OAS are of comparable magnitudes, even when π_ρ is visibly greater than π_τ. As we discuss in the next section, the potential value plunge for discount MBS caused by slower turnover rate can be too deep to bear.

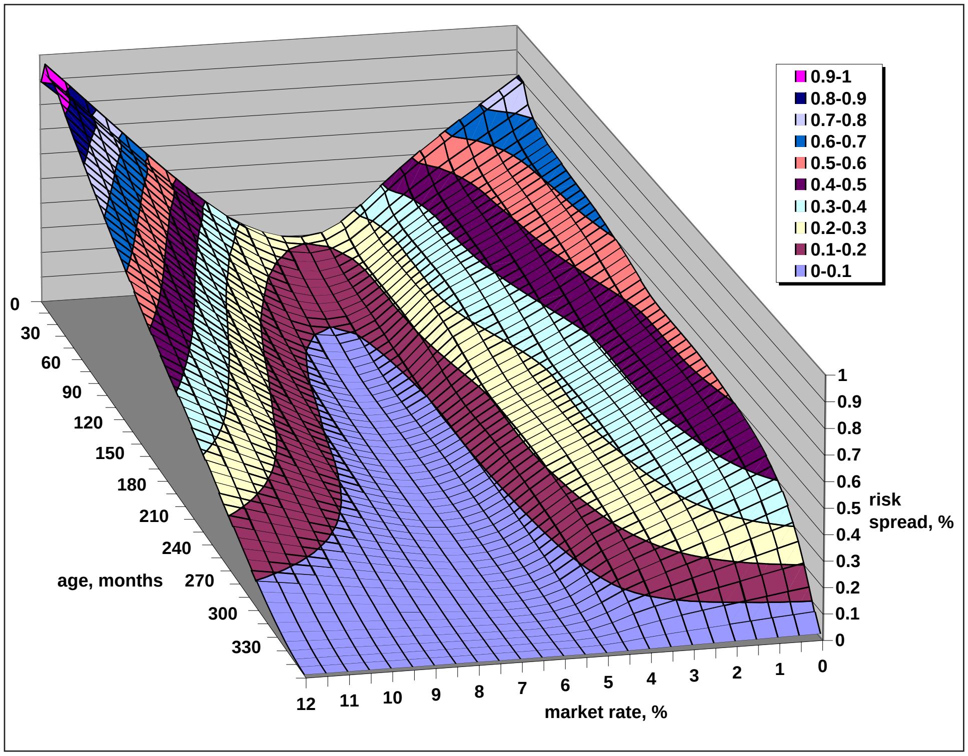
Prepay Risk Map

The prepayment risk spread, a key periodic return measure arising in the course of valuation under the prOAS method (see formula 2), can be shown for either the active or passive component in the form of a three-dimensional graph (Figure 4). As expected, the risk and related spread compensation vanish as the MBS approaches maturity. Generally, a near-par MBS is the least risky since it has no premium or discount at stake, i.e. in the trough of the chart in Figure 4. As rates rise or drop, the risk grows. Surprisingly enough, carrying the turnover risk can be as (or even more) risky as carrying the refinancing risk. Indeed, if rates drop, prepayments surge and the shortened mortgage life limits potential price uncertainty. After all, agency pass-throughs rarely trade above the 108-110 price range. In contrast, when rates rise, the MBS extends and its value takes a deep plunge that is constrained by and thus highly depends on the turnover rate. The resultant price range can vary widely with turnover uncertainty.

This entire risk spread measurement became possible courtesy of the APD pricing model, which allows for determining risk compensation for each investment period. We do not know of any other valuation model (including the Bear Stearns PORC model (see Cohler, Feldman, and Lancaster [1997]) that is conceptually capable of producing this measure.

Figure 4

Prepay Risk Map for a near-current FNCL5.0 (active part only) as of 02/21/03



The Refinancing Scare Consensus

Within the traditional OAS valuation, either price or OAS should be given as input. Under the prOAS valuation, the role of OAS is performed by prOAS, a better-defined measure. The entire goal of prOAS pricing is to eliminate differences in OAS among instruments that are exposed to prepayment risk differently. As we asserted earlier, the prOAS measure should value agency MBS flat to agency debentures. Therefore, once risk factors are given their stochastic specifications (volatility and mean reversion) and the price of risk constants are determined, we can value any agency MBS or its IO and PO strips much like swaptions, i.e. looking at the benchmark rate and volatility structure, but with no prior knowledge of traditional OAS.

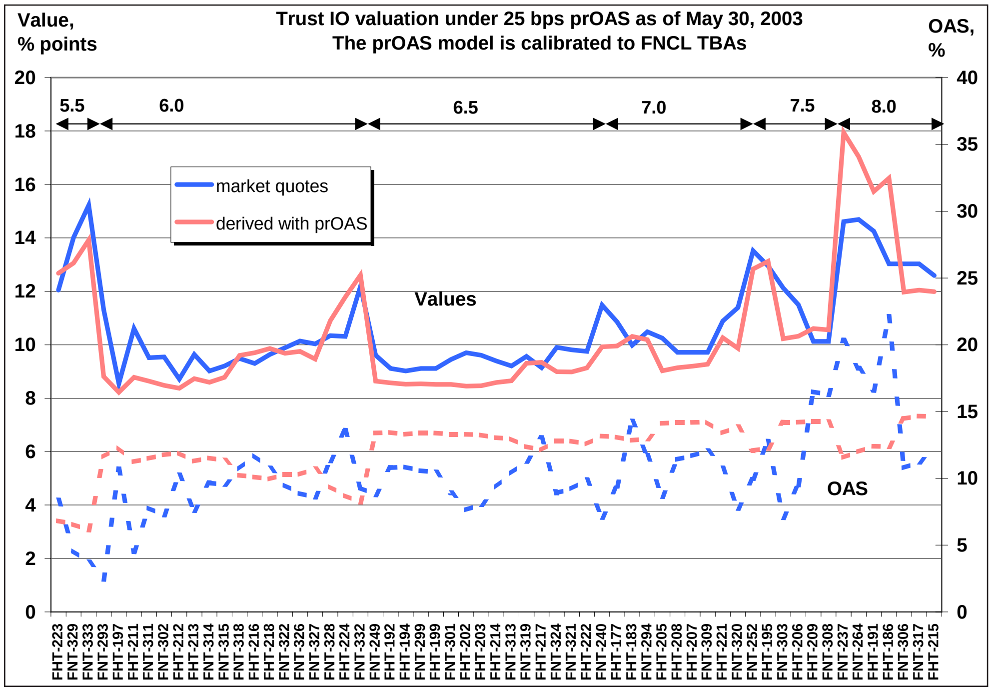
Figure 5 exhibits valuation results for agency Trust IOs using prices of risk constants, a large π_ρ and a near zero π_τ, obtained from the calibration to Fannie Mae TBAs on May 30, 2003. The application of the prOAS method first leads to values[3] which are then converted into conventional OAS measures.

[3]To account for liquidity difference between IOs and TBAs, we applied 25 basis points prOAS.

The prOAS model explains IO "cheapness" (therefore, PO "richness") naturally and correctly predicts OAS level of (and above) a thousand basis points. Since POs stripped off premium pools should be looked at as hedges against refinancing risk, they must be traded "rich" according to the APT. Our prOAS model successfully confirms this fact, having virtually all OAS for Trust POs deep in negative territory (not drawn in Figure 5). Results in Figure 5 also provide some degree of confidence in managers of mortgage servicing rights (MSR) portfolios (not actively traded or frequently quoted) — they can employ the prOAS measure to better assess the risk of their portfolios. MSRs will be discussed in more detail below.

Figure 5

Valuation of Trust IOs on 05/30/03



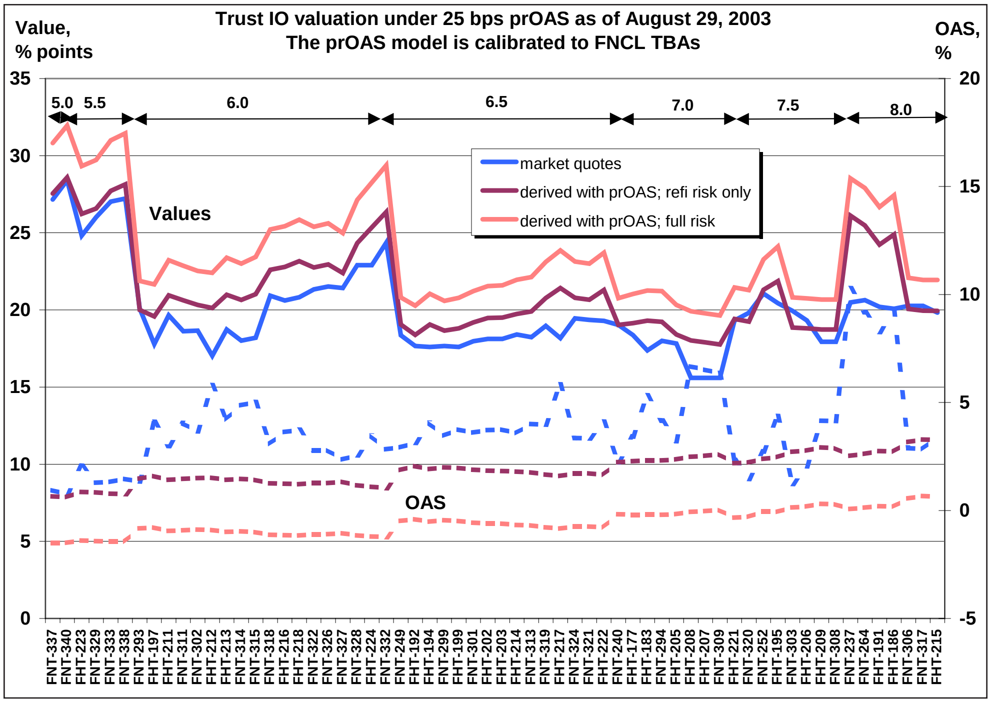
As we showed in the historical risk chart (Figure 3), on May 30, 2003, the entire risk perception was evolving out of a refinancing scare. Figure 5 illustrates that both the TBA market and the Trust IO market agree with one another when incorporating this risk into pricing.

MISSING: The Power of the Turnover Hedge

In the summer months of 2003, rates rose sharply, pushing lower-coupon MBS (4.5s and 5s) into discount territory. Figure 3 confirms that by the end of that summer the refinancing fear cooled off, making room for turnover concerns. As we explained, slower-than-modeled turnover results in a loss for a discount MBS holder. It's not surprising that the price for turnover risk, virtually non-existent in May 2003, became sizable (Figure 3). What if we apply prices of risk calibrated to the August 29, 2003 TBA market to value Trust IOs? Figure 6 below shows two stages in application of the prOAS model: valuation with refinancing-risk only and valuation with the total risk.

Figure 6

Valuation of Trust IOs on 08/29/03



Comparing the blue lines (market prices and related OAS) with the purple lines (prOAS model with refinancing risk only), we see that the single-risk-factor prOAS model just slightly overstates values relative to the actual market. It shows directionally correct OAS tendency (tighter spread for discount IOs, wider for premiums) and magnitude. We could even have brought this valuation exercise closer to the actual market quotes if we had extended the single-risk-factor view to turnover as well, assuming that IO investors were provided incremental returns for turnover risk too.

Disaster strikes when we add the true turnover risk calibrated to the TBAs (orange lines). Since IOs can be used as hedges against turnover risk, the APT says their return should be penalized, not rewarded. An almost constant 250 basis point OAS reduction is seen as the result. The actual IO market does not seem to appreciate this theory. Price quotes are much lower than the full two-risk-factor prOAS model suggests they should have been. According to the APT, such mispricing should lead to an opportunity to construct a fully hedged, risk-free portfolio that earns more than the risk-free rate. It is easy to implement this theory by simply combining pass-throughs and IOs. The prOAS model automatically measures sensitivities to the risk factors as part of the "risk accounting" process so that the two-factor delta-hedging can directly employ them.

Our analysis, if correct, proves that there exists both a theoretical and practical opportunity to create a dynamically hedged mortgage fund that is prepay-neutral and earns an excess return over funding rates.

Calibration to Trust IOs & Consideration of MSR

The theory of risk-and-option-adjusted valuation is directly applicable to mortgage servicing rights (MSR), which can be thought of as an extension of IOs. As such, they will benefit from observations and analyses performed for the Trust IO market. In fact, using this market as the prOAS calibration benchmark is a natural application of our approach to MSRs. As demonstrated, the IO market views prepayment risk as a single-dimensional risk where acceleration is the feared direction regardless of the source. We will next discuss how calibration to IOs may also involve tuning the burnout parameter ψ_0. Calibrating the prOAS model accordingly, the user can proceed to value an MSR using the actual servicing cash flows, rather than a simple IO strip. Computed Greeks (sensitivities), a primary concern of risk managers, will be closer to those observed in the IO market than the ones derived from a standard, constant OAS methodology (see Figure 7).

Consideration of Other Prepay Risk Factors

Burnout

Results in Figures 5 and 6 lead to another interesting observation. The single-risk-factor prOAS model did a good job of valuation, except for very seasoned 8% pools (FNT-237, FNT-264, etc). Perhaps the burnout model driven largely by parameter ψ_0 (initial active portion of the pool, see Part I) represents another source of model risk. Indeed, a premium MBS investor faces losses if the pool turns out to be less burnt than a physical prepay model predicts. Therefore, a risk-neutral value of ψ_0 should be somewhat higher than the historical one.

Optimal risk-neutral tuning of ψ_0 is difficult when working with TBA MBS since they generally reflect relatively new instruments. Two mathematical sides of refinancing risk, the overall multiple and the burnout, essentially overlap, making one of them redundant. This is not the case with the Trust

IOs market where all seasoning stages are represented. To make this observation evident, let us consider valuation of the FNT-264 pool. This is an 8%, 9-year old Fannie Mae pool that has gone through several historical refinancing waves. Starting with ψ_0 = 90% and using the APD analytics, we find ψ = 20% in May 2003, leading to a generous theoretical value of 17.04 (the actual market being at just 14.68). However, if we boost ψ_0 to 95%, then ψ comes in at 34.5% and the theoretical price drops down to 14.83, much closer to the actual one. This example, of course, is purely illustrative — once we change ψ_0, we must recalibrate the prices of risk. However, the point is intuitively clear: a risk-neutral tuning of ψ_0 should reduce the value of old pools relative to new pools.

The S-curve Slide

Can a historical prepayment model be biased when used for forecasting speeds? Perhaps it can if it does not reflect systematic enhancements in regulation that make refinancing hurdles lower. Investors expect the refinancing process to ease, which will trigger refinancing decisions with less of a rate incentive. The S-like refinancing curve that simulates the empirical refinancing speed as a function of rate incentive has been sliding to the left, and it may continue this trend in the future.

As we explained in Part II of this sequel (Levin [Mar. 2004]), there is no mathematical difference between accounting for the risk and an expectation bias. Both contribute identically to the pricing equation as well as to the risk-neutral transformation of a physical prepay model. For example, if our calibration process had included the slide (horizontal position of the S-curve) as another factor of risk, we probably would have seen the S-curve move to the left. This optimal transformation can be attributed to both fear and the expectation of more efficient refinancing than what we saw in the historical model. Much like in the case of burnout, using refinancing multiple ρ as the main risk factor often tends to overwhelm other details of the refinancing model, including the slide. However, an inverse OAS (coupon) profile for TBAs typically points to the existence of the slide risk.

Valuation adjusted for prepayment risk leads to different rate sensitivity than traditional valuation. Intuitively, premium pass-throughs become less rate-sensitive because their risky spread "absorbs" interest rate moves following the prepay option moneyness. Indeed, any rate fall elevates the refinancing risk, thereby inflating the traditional OAS; any rate rise reduces the risk and compresses the OAS. Since discount MBS react inversely, they are more rate-sensitive under the prOAS method than under the constant-OAS risk assessment. A flat OAS profile for the current-coupon to cuspy-premium TBA sector seen in Figure 2 suggests that the constant-OAS valuation is a valid method for assessing their rate sensitivity. All these findings (confirmed in Tables 1A, B) can be even more easily explained by the equivalent transition from the actual to risk-neutral prepayment model; faster for premiums and slower for discounts.

Table 1

From OAS to prOAS: TBA Duration difference

A. Market as of May 30, 2003 (MTGEFNCL = 4.397%)

Method	FN4.5	FN5.0	FN5.5	FN6.0	FN6.5	FN7.0	FN7.5	FN8.0
OAS	4.10	2.90	1.96	1.71	1.67	1.59	1.83	2.12
prOAS	3.42	2.03	1.07	0.77	0.80	0.90	1.11	1.28
Difference	-0.68	-0.83	-0.89	-0.94	-0.87	-0.69	-0.72	-0.84

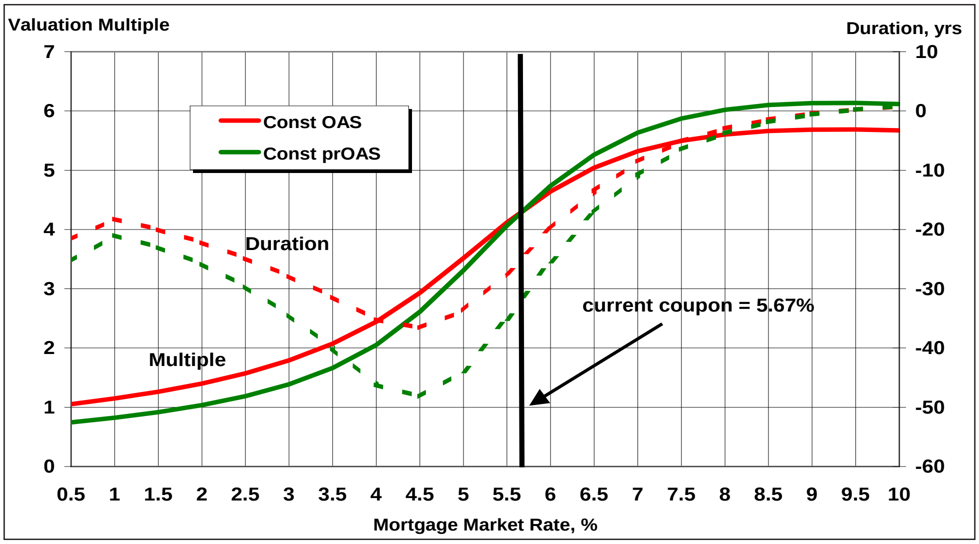
B. Market as of August 29, 2003 (MTGEFNCL = 5.670%)

Method	FN4.5	FN5.0	FN5.5	FN6.0	FN6.5	FN7.0	FN7.5	FN8.0
OAS	5.78	5.18	4.40	3.34	2.48	2.07	2.03	1.90
prOAS	6.11	5.44	4.54	3.32	2.35	1.98	1.81	1.68
Difference	+0.33	+0.26	+0.14	-0.02	-0.13	-0.09	-0.22	-0.22

In Figure 7 we compare valuation profiles for an MSR stripped off a 6.5% GWAC, slight premium pool. Unlike our previous exercises, this time we calibrate the prOAS model to Trust IOs, not TBAs. We see that constant-OAS valuation systematically understates rate sensitivity for all rate levels; the difference is significant, even for current rates. This implies that MSR managers would under-hedge using a traditional OAS approach.

Figure 7

Valuation of a 6.5% GWAC MSR under 25 bps prOAS as of 08/29/2003. The prOAS model is calibrated to Trust IOs



The prOAS valuation approach introduced and analyzed in this sequel is a well-defined extension of the traditional OAS method that draws its roots from the Arbitrage Pricing Theory. It successfully explains many phenomena of the MBS market, such as OAS variability among MBS coupons and instrument types, deviation of practical durations from theoretical, etc. At the same time, the method points to some cavities in the mortgage market that reveal inefficiencies and possible arbitrage. Two main uncovered anomalies, missed hedging power of the IOs against the turnover risk, and exaggerated dynamics of risk prices, allow rigorous investors to construct prepay-risk-neutral MBS portfolios that earn excess returns.

The prOAS valuation method is fully implemented on a backward inducting lattice and integrated with the AD&Co. valuation functions library. AD&Co.'s traditional prepayment models are being converted into the active-passive format, making them eligible for backward valuation. We also make risk-neutral versions of our prepayment models available for both backward valuation and Monte-Carlo random sampling that is applicable to CMOs.

With the recent release of additional agency pool information (loan size, geographical composition, occupancy types, quartiles, etc.) it has become possible to value MBS more accurately. The prOAS method powered by the backward inducting APD scheme is a fast and rigorous way to value this universe — addressing such issues as the worst TBA selection, specific pool pay-up relative to TBAs and to other pools, etc. We also see exciting opportunities to help MSR managers who often need guidance through the myriad of loans and groups with no price or OAS quotes available. They are perfectly poised to take full advantage of the speed and rigor of the prOAS method.

■

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About the Author

Alex Levin *Alex Levin is a Senior Quantitative Developer/Consultant responsible for the research and development of analytical models for mortgages and other financial instruments and related consulting work. Alex has developed a suite of interest rate models that can be calibrated to swap rates and a swaption volatility matrix. He also leads Andrew Davidson & Co., Inc.'s efforts in developing new efficient valuation models, including the Active-Passive Decomposition (APD) mortgage model facilitated by a backward induction OAS pricing. Alex carries out research on prepay risk-adjusted valuation that allows for objective pricing with or without an OAS input.*

Until March of 2002, Alex was a Senior Vice President and Director of Treasury Research and Analytics at The Dime Bancorp (the Dime) in New York. At the Dime, he was in charge of developing efficient numerical and analytical tools for pricing and modeling mortgages, options, deposits, and other complex term-structure-contingent derivatives, risk measurement and management. He has authored Mortgage Solutions, Deposit Solutions and Option Solutions, the Dime's proprietary pricing systems that were *intensively used for both security trading and risk assessment. In addition, he was regularly involved in measuring market risk (including a counter-party value-based risk) and hedging positions in various lines of the banking business.*

Prior to his employ at the Dime, Alex taught at The City College of NY and worked at Ryan Labs, a fixed income research and money management company.

Alex is a regular speaker at the Mathematical Finance Seminar (NYU, Courant Institute), AD&Co client conferences, and has published a number of papers. He holds an M.S. in Applied Mathematics from Naval Engineering Institute, Leningrad and a Ph.D. in Control and Dynamic Systems from Leningrad State University.

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ANDREW DAVIDSON & CO., INC.
520 BROADWAY, EIGHTH FLOOR
NEW YORK, NY 10012
212•274•9075
FAX 212•274•0545
E-MAIL mail@ad-co.com

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